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The numbers contain eight articles in English; five of them are mathematical.

A new mathematical periodical under the editorship of Doctor Ginsburg is soon to be published by Jeshiva College, New York City. In it the history of mathematics and bibliography are to receive special attention.

January 26, 1932.

A DISCUSSION OF THE GENERAL LINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER, $d^2y/dx^2+P\,dy/dx+Qy=R$, IN WHICH P, Q, AND R ARE FUNCTIONS OF x.

By ROLAND SCHAFFERT, University of Cincinnati

Various methods have been devised to obtain solutions of special forms of the above equation, and many of these special forms lead to interesting results which are of practical significance.

In the first part of the present paper several forms of this equation, heretofore not discussed in the various methods for obtaining solutions of special forms, are integrated.

The second part contains a discussion of the general equation, in which it is shown that its solution may be made to depend upon the solution of a differential equation of the first order.

Part I

If we substitute $y = e^{\int a\,dx}$, where a is a constant, in the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0,$$

we get

$$e^{\int a\,dx}(a^2 + Pa + Q) = 0.$$

From this we obtain

$$a = \tfrac{1}{2}\left[-P \pm (P^2 - 4Q)^{1/2}\right].$$

In the equation,

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R, \tag{1}$$

write

$$y = ue^{\frac{1}{2}\int[-P\pm(P^2-4Q)^{1/2}]\,dx} \tag{2}$$

where u is a function of x to be determined. Equation (1) then becomes

$$(3)\qquad \frac{d^2u}{dx^2} + (P^2 - 4Q)^{1/2}\frac{du}{dx} + \frac{1}{2}\left[-\frac{dP}{dx} \pm \frac{d}{dx}(P^2 - 4Q)^{1/2}\right] u = Re^{-\frac{1}{2}\int[-P\pm(P^2-4Q)^{1/2}]dx}.$$

Now, if

$$(4)\qquad -\frac{1}{2}\frac{dP}{dx} \pm \frac{1}{2}\frac{d}{dx}(P^2 - 4Q)^{1/2} = 0,$$

equation (3) can at once be solved.
Equation (4) may be written $dP = \pm d(P^2-4Q)^{1/2}$ and therefore

$$P = \pm(P^2-4Q)^{1/2} \pm A.$$

Solving for Q, we get $-\frac{1}{4}A^2 \pm \frac{1}{2}AP$. Letting $\frac{1}{2}A = n$, we have $Q = -n^2 \pm nP$.
Substituting this value of Q in equation (3), we have

$$\frac{d^2u}{dx^2} \pm (P \mp 2n)\frac{du}{dx} = Re^{-\frac{1}{2}\int[-P\pm(P\pm 2n)]dx}.$$

The first integral of this is

$$\frac{du}{dx} = Be^{-\int\pm(P\mp 2n)dx} + e^{-\int\pm(P\mp 2n)dx}\int Re^{\frac{1}{2}\int[P\pm(P\mp 2n)]dx}dx$$

and therefore

$$u = C + B\int e^{-\int\pm(P\mp 2n)dx}dx + \int\left[e^{-\int\pm(P\mp 2n)dx}\int Re^{\frac{1}{2}\int[P\pm(P\mp 2n)]dx}dx\right]dx.$$

Write this value of u in equation (2), and we get

$$y = Ce^{\frac{1}{2}\int[-P(P\mp 2n)]dx} + Be^{\frac{1}{2}\int[-P\pm(P\mp 2n)]dx}\int e^{-\int\pm(P\mp 2n)dx}dx + e^{\frac{1}{2}\int[-P\pm(P\mp 2n)]dx}\int\left[e^{-\int\pm(P\mp 2n)dx}\int Re^{\frac{1}{2}\int[P\pm(P\mp 2n)]dx}dx\right]dx,$$

which is a general solution of equation (1), where

$$-\frac{1}{2}\frac{dP}{dx} \pm \frac{1}{2}\frac{d}{dx}(P^2 - 4Q)^{1/2} = 0.$$

The form to which this may readily be applied is

$$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} - n^2y \pm nf(x)y = R(x).$$

Example 1:

$$\frac{d^2y}{dt^2} + t\frac{dy}{dt} - n^2y + nty = \sin(pt).$$

The solution of this is

$$y = Ce^{-nt} + B\frac{e^{nt-\frac{1}{2}t^2}}{2n-t} + e^{-nt}\int\left[e^{2n-\frac{1}{2}t^2}\int \sin(pt)e^{\frac{1}{2}t^2-nt}dt\right]dt.$$

Part II

In the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \tag{1}$$

let $Q = t+v$, where t and v are functions of x.
Equation (1) becomes

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + (t+v)y = R. \tag{3}$$

If we put

$$= ue^{\frac{1}{2}\int[-P\pm(P^2-4v)^{1/2}]dx}, \tag{4}$$

equation (3) becomes

$$\frac{d^2u}{dx^2} \pm (P^2-4v)^{1/2}\frac{du}{dx} + \frac{1}{2}\left[-\frac{dP}{dx} \pm \frac{d}{dx}(P^2-4v)^{1/2} + t\right]u = Re^{-\frac{1}{2}\int[-P\pm(P^2-4v)^{1/2}]dx}. \tag{5}$$

Thus far, the only limitation placed upon t and v is that their sum must be equal to Q. Hence we may choose any relation between them. Let this relation be

$$t = \frac{1}{2}\frac{dP}{dx} \mp \frac{1}{2}\frac{d}{dx}(P^2-4v)^{1/2}. \tag{6}$$

The third term on the left-hand side then vanishes, and equation (5) becomes

$$\frac{d^2u}{dx^2} \pm (P^2-4v)^{1/2}\frac{du}{dx} = Re^{-\frac{1}{2}\int[-P\pm(P^2-4v)^{1/2}]dx},$$

or, putting $(P^2-4v)^{\frac{1}{2}} = \theta$, this becomes

$$\frac{d^2u}{dx^2} \pm \theta\frac{du}{dx} = Re^{-\frac{1}{2}\int(-P\pm\theta)dx}. \tag{7}$$

This being an equation of the linear form in du/dx, when solved, it is

$$\frac{du}{dx} = Ae^{\pm\int\theta dx} + e^{\pm\int\theta dx}\int Re^{\mp\int(P\pm\theta)dx}dx,$$

It follows that

$$u = B + A\int e^{\mp\int\theta dx}dx + \int\left[e^{\pm\theta dx}Re^{\mp\int(P\pm\theta)dx}dx\right]dx;$$

so that

$$ye^{-\frac{1}{2}\int(-P\pm\theta)dx} = B + A\int e^{\mp\int\theta dx}dx + \int\left[e^{\mp\int\theta dx}Re^{\frac{1}{2}\int(P\pm\theta)dx}dx\right]dx$$

or

$$(8)\quad y = Be^{\frac{1}{2}\int(-P\pm\theta)dx} + Ae^{\frac{1}{2}\int(-P\pm\theta)dx}\int e^{\mp\int\theta dx}dx + e^{\frac{1}{2}\int(-P\pm\theta)dx}\int\left[e^{\mp\theta\int dx}\int Re^{\frac{1}{2}\int(P\pm\theta)dx}dx\right]dx.$$

It remains, then, to get a value for θ.
By combining equation (2) and equation (6), we get

$$\frac{1}{2}\frac{dP}{dx} \mp \frac{1}{2}\frac{d}{dx}(P^2 - 4v)^{1/2} = Q - v;$$

or

$$\frac{1}{2}\frac{dP}{dx} \mp \frac{1}{2}\frac{d\theta}{dx} = Q - \tfrac{1}{4}P^2 + \tfrac{1}{4}\theta^2,$$

so that[1]

$$(9)\qquad \pm\frac{d\theta}{dx} - \tfrac{1}{2}\theta^2 = \left(2Q - \tfrac{1}{2}P^2 - \frac{dP}{dx}\right).$$

Any solution whatever of this equation, giving a value for θ, permits a general solution of equation (1).
For example, if $Q = dP/dx$, a solution of equation (9) in this case is $\theta = P$. It is then evident that differential equations of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + \frac{dP}{dx}y = R$$

can be solved.

[1] Observe that the right-hand side of this expression is $2I$, where I is the invariant as defined in Forsyth's *Differential Equations*, Sixth edition, p. 105.

Example 2:

$$\frac{d^2y}{dx^2} + x^n\frac{dy}{dx} + nx^{n-1}y = \sin px.$$

We have here

$$\pm\frac{d\theta}{dx} - \frac{\theta^2}{2} = nx^{n-1} - \frac{x^{2n}}{2},$$

so that $\theta = x^n$. The primitive, then, is given by

$$y = Be^{-x^{n+1}/n+1} + Ae^{-x^{n+1}/n+1}\int e^{x^{n+1}/n+1}dx + e^{-x^{n+1}/n+1}\int\left[e^{x^{n+1}/n+1}\int\sin pxdx\right]dx.$$

Also, if $Q = \frac{1}{4}P^2 + \frac{1}{2}dP/dx$ equation (9) becomes $d\theta/dx - \frac{1}{2}\theta^2 = 0$.
Its solution is $\theta = \pm 2/x$.
It is thus seen that equations of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + \left(\frac{P^2}{4} + \frac{dP}{dx}\right)y = R$$

may be solved.

Example 3:

$$\frac{d^2y}{dx^2} + x^n\frac{dy}{dx} + \left(\frac{x^{2n}}{4} + nx^{n-1}\right)y = f(x).$$

Since $\theta = \pm 2/x$ the primitive of this is

$$y = Be^{\frac{1}{2}\int(2x^{-1}-x^n)dx} + Ae^{\frac{1}{2}\int(2x^{-1}-x^n)dx}\int e^{-\int 2x^{-1}dx}dx$$
$$+ e^{\frac{1}{2}\int(2x^{-1}-x^n)dx}\int\left[e^{-\int 2x^{-1}dx}\int f(x)e^{\frac{1}{2}\int(2x^{-1}+x^n)dx}dx\right]dx.$$

The equation for θ may in some cases become integrable by suitable substitutions. For example, if we let $\theta = -(1/z)(dz/dx)$, equation (9) becomes[1]

$$(d^2z/dx^2) + 4Iz = 0, \tag{10}$$

where $2I = 2Q - \frac{1}{2}P^2 - (dP/dx)$.

Evidently, if solutions of equation (10) are possible, we get values for θ satisfying the general solution of the original equation. For example, if $2I$ = a constant $= n$, then

$$z = e^{\pm ix\sqrt{2n}} \text{ and } \theta = i\sqrt{2n}.$$

In this case, $Q = \frac{1}{2}n + \frac{1}{4}P^2 + \frac{1}{2}dP/dx$. Hence the equations of the form

$$d^2y/dx^2 + Pdy/dx + [\tfrac{1}{2}n + \tfrac{1}{4}P^2 + \tfrac{1}{2}dP/dx]y = f(x)$$

can be solved.

[1] This is the normal form given by Forsyth, p. 105.

Example 4:

$$\frac{d^2y}{dx^2} + x^m\frac{dy}{dx} + \left[\tfrac{1}{2}n + \tfrac{1}{4}x + \tfrac{1}{2}mx^{m-1}\right]y = f(x).$$

Since $\theta = i\sqrt{(2n)}$ the primitive is

$$y = Be^{\frac{1}{2}\int[-x^m \pm i\sqrt{(2n)}]dx} + Ae^{\frac{1}{2}\int[-x^m \pm i\sqrt{(2n)}]dx}\int e^{\mp\int i\sqrt{(2n)}dx}dx$$

$$+ e^{\frac{1}{2}\int[-x^m \pm i\sqrt{(2n)}]dx}\int\left[e^{\mp\int i\sqrt{(2n)}dx}\int f(x)e^{\frac{1}{2}\int[x^m \pm i\sqrt{(2n)}]dx}dx\right]dx.$$

In general, if any solution whatever of

$$\pm\,(d\theta/dx) - \tfrac{1}{2}\theta^2 = 2Q - \tfrac{1}{2}P^2 - (dP/dx)$$

can be obtained, the general solution of the original differential equation is at once possible.

A Supplementary Note by Harris Hancock

The differential equation

$$\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = 0$$

may be written symbolically $f(D)y = 0$. (See Forsyth, *Differential Equations*, 4th Edition, page 66.) In this equation write $y = e^{kx}u(x)$, where k is a constant and $u(x)$ a function of x to be determined; and use the theorem given by Forsyth, p. 58. It is seen that

$$(1)\qquad f(D)y = f(D)\{e^{kx}\cdot u(x)\} = e^{kx}f(D+k)\{u(x)\}$$

$$= e^{kx}\left[f(k) + \frac{f'(k)}{1!}D + \cdots + \frac{f^n(k)}{n!}D^n\right]\{u(x)\},$$

the expansion terminating, as f is a polynomial of finite degree n.

If the equation $f(k) = 0$ has only simple roots, we give constant values to $u(x)$. If there are double roots, so that $f'(k) = 0$, we take $u(x) = A_1x + A$; if there is a triple root, so that $f''(k)$ is also zero, we take $u(x) = A_2x + A_1x + A$, etc. It is thus seen that in all cases the right-hand side of (1) is made zero. The complementary function may be thus determined without requiring the arbitrary constants, A_1 and A, say, to "become infinite in such a way" that the coefficients of the powers of x remain finite.

The same method is applicable to solutions of equations of the form

$$x^n\frac{d^ny}{dx^n} + A_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + A_{n-1}x\frac{dy}{dx} + A_ny = 0.$$

The above methods of solution were found by my colleague, Professor Louis Brand, when he was a student in one of my classes in the University of Cincinnati.

ON THE ISODYNAMIC POINTS OF FOUR SPHERES

By NATHAN ALTSHILLER-COURT, University of Oklahoma

1. Theorem. *The two inverses, with respect to two given spheres (circles), of a point on their sphere (circle) of similitude are symmetrical with respect to the radical plane (axis) of the given spheres (circles).*

Let P, Q, be the inverse points of the point W with respect to two given spheres (A), (B), having A, B for centers, and a, b for squares of their radii. We have then

$$AP \cdot AW = a, \quad BQ \cdot BW = b.$$

If, further, W is a point on the sphere of similitude of (A), (B), we have also

$$WA^2 : WB^2 = a : b,$$

hence

$$AP : BQ = AW : BW.$$

This proportion shows that the lines PQ, AB are parallel.

The pairs of points W, P; W, Q, being inverse with respect to (A), (B), any sphere (O) passing through W, P, Q, is orthogonal to both (A) and (B), hence the plane (R) through the center O of (O) and perpendicular to AB is the radical plane of the spheres (A), (B), and since PQ is parallel to AB the plane (R) is also perpendicular to PQ. But the plane through O perpendicular to PQ is clearly the mediator (i.e., the perpendicular bisecting plane) of the segment PQ, hence the proposition.

2. *Two imaginary spheres (A), (B), with real centers A, B, but the squares a, b of whose radii are negative, have a real sphere of similitude, namely the locus of the point W such that*

$$WA^2 : WB^2 = a : b.$$

The above proposition is valid for two such spheres, for in the proof no use is made of points on either of the two given spheres.

3. If the point W (1) describes the sphere of similitude (S) of the given spheres (A), (B), we have the theorem: *If the sphere of similitude of two given spheres is inverted with respect to the given spheres as spheres of inversion, the two inverse spheres thus obtained are equal and symmetrical with respect to the radical plane of the given spheres.*

It may further be observed that the centers of the two inverses (S_a), (S_b) of (S) with respect to (A), (B) lie on the line of centers of (A), (B). For (S) is coaxal with (A), (B), and (S_a) is coaxal with (A) and (S). Similarly for (S_b).

4. Definitions. Four given spheres (A), (B), (C), (D), taken in pairs have six spheres of similitude. These six spheres have two points W, W' in common, real or conjugate imaginary.[1] The points W, W' are the limiting points of the

[1] J. L. Coolidge. A treatise on the circle and the sphere, p. 244, Theorem 55. Oxford, 1916.

coaxal pencil of spheres determined by the orthogonal sphere (R) of the four given spheres and the sphere (O) passing through the centers A, B, C, D, of these four spheres. The points W, W' are thus perfectly determined, independently of the six spheres of similitude, and may be considered as two points connected directly with the four given spheres. For want of a better term the points W, W' may be called the "isodynamic points" of the four given spheres.

The radical plane of the two spheres (R), (O) is sometimes referred to as the "Newtonian plane" of the spheres (A), (B), (C), (D). This plane contains the centers of the six spheres of similitude of the given spheres.[1]

In what follows it is assumed that the isodynamic points are real, which implies that the four given spheres are either all real or all imaginary.

5. Theorem. *The four inverses, with respect to four given spheres, of an isodynamic point of these spheres, determine a sphere concentric with the orthogonal sphere of the given spheres.*

With the same notations as in (4), let P, Q, be the inverses of W with respect to (A), (B). The lines PQ, AB, are parallel (1), hence the mid-point L of PQ and the mid-point M of the line of centers AB of (A), (B), are collinear with W, and we have

$$WP:WA = WL:WM.$$

We observe also that the mediators of the segments PQ, AB are parallel.

The mediator of AB contains the center O of the sphere (O) (4), and the mediator of PQ coincides with the radical plane of the spheres (A), (B) (1), and therefore contains the center R of the sphere (R) (4). Furthermore the points R, O, are collinear with W. Hence OM is parallel to RL, and we have

$$WL:WM = WR:WO$$

hence

$$WP:WA = WR:WO.$$

This proportion shows that the lines OA, RP are parallel, and we have

$$RP:OA = WR:WO.$$

The last three terms of this proportion do not depend upon the sphere (A) considered, hence RP is constant, which proves the proposition.

6. Let (R') denote the sphere determined by the four inverses of W with respect to the spheres (A), (B), (C), (D). It follows from the proof of the preceding proposition (5) that the point W divides the line of centers OR of the two spheres (O), (R') in the ratio of the radii of these spheres, hence the isodynamic point W is a center of similitude of the two spheres (O) and (R').

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[1] Mathesis, 1926, pp. 467–8.

7. The last two propositions (5, 6) have their analogues in relation to three circles in the plane. The reader may find it of interest to formulate these propositions. Their proofs differ but little from those given above.

8. Let a be the radius of the sphere (A) (5). The points W, P being inverse with respect to (A), we have

$$AP \cdot AW = a^2,$$

and from the similar triangles WOA, WRP we have

$$AW : AP = OW : OR$$

hence

$$(a) \qquad AW^2 : a^2 = OW : OR.$$

Since the right hand side of this proportion does not depend upon the sphere (A) considered we have here a proof of the known proposition: *The distances of an isodynamic point of four spheres from the centers of these spheres bear a constant ratio to the radii of the respective spheres.* Furthermore the proportion (a) furnishes a geometric meaning for the constant involved in this proposition.

9. Applying Stewart's theorem to the triangle WOA (8), we have, both in magnitude and in sign,

$$OW \cdot AR^2 + WR \cdot OA^2 + RO \cdot AW^2 + OW \cdot WR \cdot RC = 0.$$

Let a, r be the radii of the spheres (A), (O) (5), and s the square of the radius of the orthogonal sphere (R) of the four given spheres (A), (B), (C), (D), (5).

The two spheres (A), (R), being orthogonal, we have

$$AR^2 = a^2 + s.$$

Further, if we put (8)

$$AW^2 : a^2 = OW : OR = 1 : k$$

we have

$$WR = (k - 1)OW, \; RO = -k \cdot OW, \; RO \cdot AW^2 = -OW \cdot a^2.$$

Substituting these values in the Stewart formula above, we obtain, after simplification,

$$(b) \qquad s = (1 - k)(r^2 - k \cdot OW^2).$$

Thus if the positions of the three points R, O, W, are known, and either s or r^2 is given, the other radius is determined.

10. The foregoing propositions yield interesting resulst when applied to the tetrahedron. By way of introduction, I shall state here, for the convenience of

the reader, the following propositions, since they can hardly be assumed to be "well known," nor can they be dismissed with a ready reference. They are due to Gaspar Monge (1746–1818).[1]

(*a*). Given a tetrahedron $(T)=ABCD$, if through the three pairs of opposite edges of (T) we pass three pairs of parallel planes we obtain a parallelepiped (P) which is said to be "circumscribed" about (T). Two opposite edges of (T) are two non-parallel diagonals of a pair of opposite faces of (P).

(*b*). Let A', B', C', D', be the vertices of (P) diagonally opposite A, B, C, D, respectively. The diagonal AA' of (P) meets the face BCD of (T) in the centroid G_a of the triangle BCD, and $AG_a=\frac{2}{3}AA'$. Thus the four diagonals of (P) contain the medians AG_a, BG_b, CG_c, DG_d, of (T), and the centroid G of (T) is the point common to the diagonals of (P).

(*c*). The six planes perpendicular to the edges of a tetrahedron (T) and passing through the mid-points of the respectively opposite edges have a point M in common, the "Monge point" of the tetrahedron (T). The six planes themselves may be referred to as the "Monge planes" of (T). The Monge point is the symmetric of the circumcenter O of (T) with respect to the centroid G of (T).

(*d*). The tetrahedron (T') determined by the points A', B', C', D' (10*b*) is also inscribed in (P). The two tetrahedrons (T), (T') may be said to be "twin" to each other. They are symmetrical with respect to their common centroid G, so that the Monge point M of (T) is the circumcenter of (T'), and the Monge planes of (T) are the mediators of the edges of (T').

11. Consider now the four spheres (A), (B), (C), (D), having for diameters the medians AG_a, BG_b, CG_c, DG_d, of the tetrahedron $(T)=ABCD$. The distance of the centroid G of (T) from the center of (A) is equal to $\frac{1}{4}AG_a$, according to Commadino's theorem, and similarly for the other three spheres. Thus the distances of G from the centers of the four spheres considered are proportional to the radii of these spheres, hence G is an isodynamic point of the four spheres (A), (B), (C), (D) (8).

The inverse of G with respect to the sphere (A) coincides with the harmonic conjugate X of G with respect to A, G_a. Now since

$$AG:GG_a = 3:1,$$

we have

$$AG_a:G_aX = 2:1,$$

i.e., X coincides with the vertex A' of the tetrahedron (T') twin to (T) (10).

Thus the four inverses of G with respect to (A), (B), (C), (D), are the vertices of (T'), hence the sphere determined by these four inverses is identical with the circumsphere of (T'). Now the circumcenter of (T') is the Monge point M of (T) (10), hence (5): *The radical center of the four spheres having for diameters the medians of a tetrahedron coincides with the Monge point of the tetrahedron.*

[1] Correspondence sur l'Ecole Polytechnique, vol. II.

12. The line of centers of the spheres (A), (B), (11) is parallel to the edge AB of (T), hence the radical plane (F) of these two spheres is perpendicular to AB. But (F) also passes through the Monge point M of (T) (11), hence (F) is a Monge plane of (T) (10c), and therefore bisects the edge CD opposite to AB on (T). This property of the radical plane of the two spheres (A), (B), has been pointed out by Prof. R. W. Genese.[1]

13. The centers of the four spheres (A), (B), (C), (D) (11) are the homothetic points of the vertices A, B, C, D, of the tetrahedron (T), the point G being the homothetic center, and the homothetic ratio being 1:3; hence the sphere (O') determined by these four centers is the homothetic of the circumsphere (O) of (T). Thus the center O' of (O') lies on the Euler line MGO of (T), and we have $GO' : GO = 1:3$; and further if r, r' are the radii of (O), (O'), we have $r' : r = 1:3$.

The square of the radius of the orthogonal sphere (M) of the four spheres considered may now be determined by the formula (b) of (9). The ratio $OW:OR$ is here to be replaced by $O'G:O'M = 1:4$, hence $k = 4$, and (b) gives for the square of the radius of (M)

$$s = (1 - 4)(r'^2 - 4.0'G^2).$$

In order to have s in terms of the elements of (T), we may write

$$s = (1 - 4)(r^2:9 - 4\cdot OG^2:9) = \tfrac{1}{3}(4\cdot OG^2 - r^2) = \tfrac{1}{3}(MO^2 - r^2).$$

Now the last parenthesis represents the power of M with respect to (O), hence: *The square of the radius of the orthogonal sphere of the four spheres having for diameters the four medians of a tetrahedron is equal to one third of the power of the Monge point of the tetrahedron with respect to the circumsphere of the tetrahedron.*

13a. The value of s may be obtained directly, without the use of formula (b). Let H be the other isodynamic point, besides G, of the four spheres (A), (B), (C), (D) (13). The two points are thus inverse with respect to both (O') and (M), hence

$$O'G\cdot O'H = r'^2, \quad MG\cdot MH = s$$

Now

$$MG = -3O'G, \quad MH = MO' + O'H = -4O'G + O'H,$$

hence

$$s = 12O'G^2 - 3r'^2.$$

From this formula and the relations

$$MO = -2\cdot OG = -6\cdot O'G, \quad r = 3r'$$

we obtain the required formula (13).

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[1] Educational Times, Reprints, vol. VI (1918), p. 57, Q. 18677.

14. In a given sphere (O) an infinite number of tetrahedrons (T) may be inscribed having for centroid a given point G, taken inside the sphere (O). All these tetrahedrons will have the same Monge point M (10c). Now the power of the fixed point M with respect to the fixed sphere (O) is obviously fixed, hence the sphere (M) (13) is fixed. It is also clear that the sphere (O') (13) is fixed. Hence: *If a tetrahedron varies so that its circumsphere and its centroid remain fixed, the spheres having for diameters the medians of the tetrahedron are orthogonal to a fixed sphere and have their centers on another fixed sphere.*

It may be observed that the sphere (M) will be real, null, or imaginary, according as the distance of the centroid G from the center O of (O) is greater than, equal to, or smaller than half the radius of (O).

15. Suppose the median AG_a of the tetrahedron $(T)=ABCD$ be divided by the point A' in the given ratio $GA':GA=u$. Let (A') be the sphere with A' as center, and $a'=v.GA$ as radius, where v is a given number. Let (B'), (C'), (D'), be the analogous spheres relative to the vertices B, C, D, of (T).

As before (11) here again the distances of the centroid G of (T) to the center of the four spheres considered are proportional to the radii of these spheres, hence G is an isodynamic point of these spheres. Furthermore the circumsphere (O) of (T) and the sphere (O') determined by the centers of the four spheres (A'), (B'), (C'), (D'), are homothetic with respect to G, hence the center O' of (O') is collinear with O and G, i.e., O' lies on the Euler line OG of (T). Now the radical center R of the four given spheres is collinear with G and O', hence: *If the medians of a tetrahedron are divided in the same ratio and the points of division are taken for centers of four spheres whose radii are proportional to the respective medians, the radical center of these four spheres lies on the Euler line of the tetrahedron.*

The special case, when the points of division coincide with the vertices of (T) and the radii are equal to the medians, has already been noticed.[1]

16. The exact position of R(15) with reference to G and O may be found by the formula (a) established before (8). We have, in the present case,

$$GA'^2:(v\cdot GA)^2 = O'G:O'R, \text{ and } GA' = u\cdot GA;$$

hence

$$O'G:O'R = u^2:v^2.$$

On the other hand we have, from the homothetic ratio of the spheres (O'), (O),

$$GO':GO = GA':GA = u,$$

hence

$$O'G = u\cdot OG$$

[1] Hoffmann's Zeitschrift, vol. 27 (1896), p. 589, Q. 1456.

and

$$O'G:O'R = O'G:(O'G + GR) = u\cdot OG:(u\cdot OG + GR) = u^2:v^2$$

hence

$$OR:OG = (v^2 - u^2 + u):u. \qquad (f)$$

This is the required formula, in magnitude and in sign.

If A' coincides with the mid-point of AG_a, and the radius of (A') is equal to one half of AG_a, we have $u=1/3$, $v=2/3$, and the formula (f) gives

$$OR:OG = 2$$

i.e., R coincides with the Monge point of (T), as has been found already (11).

17. The value of the square of the radius of the orthogonal sphere (R) of the four spheres (A'), (B'), (C'), (D') (15) may be found by the formula (b), derived before (9). The value of $1:k$ of that formula is in the present case equal to $u^2:v^2$. If r, r' are the radii of the spheres (O), (O'), we have

$$r' = ur, \text{ and } O'G = u.OG.$$

Thus the formula (b) gives for the square of the radius of (R)

$$s = (u^2 - v^2)(u^2r^2 - v^2\cdot OG^2):u^2.$$

For the special case when $u=1/3$, $v=2/3$ this formula gives

$$s = \tfrac{1}{3}(4\cdot OG^2 - r^2)$$

as was to be expected (13).

TWO PROBLEMS IN POTENTIAL THEORY

By THORNTON C. FRY, Bell Telephone Laboratories

1. *Introductory Remarks.*

"Electron guns" are electrical devices that produce a beam of electrons, all traveling as nearly as possible in the same direction. They are generally constructed on a very simple principle. The source of electrons is enclosed by a metal surface, in which a small hole is cut. When this surface is maintained at a potential higher than that of the source, the electrons are drawn toward it. Some, however, pass through the hole by virtue of their own inertia. Under suitable conditions, all the particles in this "beam" will move with substantially the same speed and in substantially the same direction.

The study of the potential field set up in such an electron gun therefore requires only the solution of Laplace's equation, with two known equipotential surfaces as boundary conditions. In this sense it belongs to the simplest type of Potential Theory problems; but the shape of the electrodes, and in particular the presence of the holes, removes it from the class of the purely trivial.

The two particular cases treated in this paper have been chosen largely because they require such radically different methods of attack. It is interesting to note that, in spite of the similarity of the problems when expressed in physical terms, neither of the methods can be used conveniently, if at all, in the solution of both problems.

2. *The Potential Field about a Slotted Cylinder.*

The first problem is that of a fine cylindrical wire placed on the axis of a slotted cylinder. In order to simplify the solution, we shall regard the wire as infinitely thin[1] (a mathematical line), and shall call the radius of the cylinder r. For convenience we shall also choose the axis of the cylinder as one coordinate axis; we know by symmetry that this coordinate will not appear in our solution. Another coordinate axis (called u) will be passed through the center of the slot, and of course the third (called v) will be normal to both. The cross-section of the system thus has the appearance shown in Fig. 1.

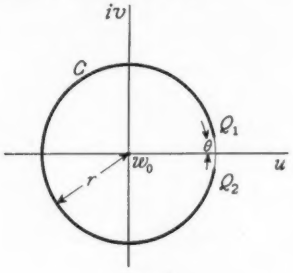


FIG. 1

Our problem, then, is to satisfy the differential equation

$$\nabla^2\phi = 0 \tag{1}$$

in two dimensions, subject to the boundary conditions, first, that[2]

$$\phi = 0, \text{ along curve } C, \tag{2}$$

and second, that ϕ shall have a logarithmic[3] infinity at the origin, which we may write

$$\lim_{(u^2+v^2)^{1/2}=0}\left[\frac{\phi(u, v)}{\log (u^2 + v^2)^{1/2}}\right] = k, \tag{3}$$

k being a finite constant, not zero.

[1] After the solution has been obtained it will be found that the equipotential surfaces in the neighborhood of this line differ but little from concentric cylinders of circular cross-section, so that the solution is an excellent approximation to the case of a wire of finite dimensions inside a cylinder of much larger radius.

[2] To give ϕ any other value Φ along this curve, it is only necessary to add the constant Φ to the solution which satisfies (2), since if (1) and (3) are satisfied by ϕ they are also satisfied by $\phi+\Phi$.

[3] The potential field produced by a charge distributed along an infinitely thin wire is known to have a logarithmic singularity. See O. D. Kellogg, *Foundations of Potential Theory*, p. 62.

These conditions are obvious enough. There is another which might be overlooked. It is, that the potential shall *not* become infinite at infinite distance from the cylinder. To violate this condition would be equivalent to supposing that the slotted cylinder C was surrounded by another (infinitely) large conductor on which additional charge was placed.

Since either the real or imaginary part of any analytic function satisfies (1), the problem may also be stated as follows: to find an analytic function $f(w)$, where $w=u+iv$, of which the real part vanishes along C and becomes logarithmically infinite at w_0. Moreover, since singularities can only occur in the presence of charge,[1] $f(w)$ must be regular throughout all space, except perhaps on C and at w_0.

Such a function can easily be found by the use of certain simple conformal transformations.

We begin with the variable $\zeta=\xi+i\eta$ and write down the analytic function

$$f = \tfrac{1}{2}k \log [(\zeta - \zeta_0)/(\zeta - \bar{\zeta}_0)], \tag{4}$$

in which ζ_0 is an arbitrary complex number and $\bar{\zeta}_0$ its conjugate. (See Fig. 2). For the time being k may also be regarded as an arbitrary constant, though we

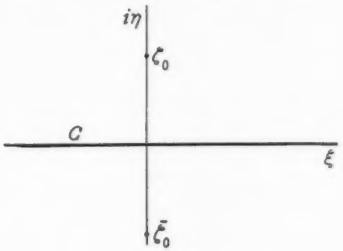


FIG. 2

shall see later on that it has been so chosen as to conform to the use of the same letter in equation (3).

It is readily seen that the real part of this function f vanishes along the entire real axis, and becomes logarithmically infinite at the point[2] ζ_0. In other words, it possesses at these places just the properties which our desired function must have on the curve C and at the point w_0 of Fig. 1; and our purpose is to construct from it a new function which will have these properties where desired.

The transformation

$$z = (\zeta^2 + a^2)^{1/2} \tag{5}$$

maps the ζ-plane on the z-plane in such a way that the real axis of the former corresponds to that part of the real z-axis for which $|x|>a$. (See Fig. 3). The

[1] See O. D. Kellogg, loc. cit., p. 139.

[2] It also becomes logarithmically singular at $\bar{\zeta}_0$, but this will cause us no difficulty, for reasons which will appear later.

remainder of the real z-axis corresponds to imaginary values of ζ; and $z=\pm a$ are branch points. The arbitrary point $\bar{\zeta}_0$ goes into an arbitrary point z_0 of Fig. 3.

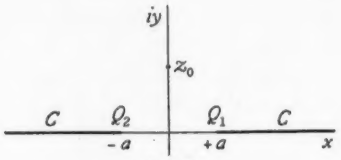


FIG. 3

Since every possible value of z may be obtained by letting ζ vary over the upper half of its plane, the physical z-plane may be said to correspond to only that part of the ζ-plane. As $\bar{\ }_0$ is thus excluded, the undesirable singularity of (4) has now disappeared.[1]

It follows then, that the function

$$f = \tfrac{1}{2}k \log \frac{(z^2 - a^2)^{1/2} - \zeta_0}{(z^2 - a^2)^{1/2} - \bar{\zeta}_0} \tag{6}$$

which we obtain by expressing (4) in terms of z, must have a real part which vanishes along the part C of the real z-axis, and which becomes logarithmically infinite at z_0.

Finally, we use the bilinear transformation

$$w = r[(i - z)/(i + z)]. \tag{7}$$

which carries all the straight lines of the z-plane into circles in the w-plane. In particular, the real axis of Fig. 3 is carried into the circle of radius r in Fig. 1; and the portion of the axis which was C in Fig. 3 therefore becomes the heavy circular arc of Fig. 1. The point z_0 of Fig. 3 also goes into some point w_0 of Fig. 1.[2]

It follows, then, that the real part of the function[3]

$$f = \tfrac{1}{2}k \log \frac{\left[-a^2 - \left(\frac{w-r}{w+r}\right)^2\right]^{1/2} - \zeta_0}{\left[-a^2 - \left(\frac{w-r}{w+r}\right)^2\right]^{1/2} - \bar{\zeta}_0}, \tag{8}$$

[1] In other words, (5) defines a multiple-valued transformation, only one branch of which is of physical significance. The singularity at ζ_0 has passed into the non-physical branch.

[2] Whether or not w_0 is at the center of the circle depends, of course, on the choice of the (as yet) arbitrary point ζ_0. In constructing the figures, however, we have placed all the arbitrary points in the positions which our subsequent discussion shows they should occupy.

[3] Since we are to use only ζ's which lie above the real axis, the radicals in (6) and (8) must all be chosen with their imaginary part positive. This leads to the rule given later on, that the *real* part of the radicals in (9), (11) and (14) must be positive.

which we obtain by rewriting (6) in terms of w, vanishes along the arc C and becomes logarithmically singular at w_0.

We must choose a and ζ_0 in such a way that $w_0 = 0$, and also so that the angular distance between the points Q_1 and Q_2 of Fig. 1 is 2θ. But if we give w_0 the value zero—that is, if we place the singularity at the center of the circle—we obtain from (7)

$$z_0 = i,$$

and hence also from (5)

$$\zeta_0 = i(a^2 + 1)^{1/2}. \tag{9}$$

These equations fix the positions of the points ζ_0 and z_0 in Figs. 2 and 3.

Similarly, the points Q_1 and Q_2 in Fig. 3 correspond to $z = \pm a$, and hence by (7) the corresponding values of w are

$$w = r[(i \mp a)/(i \pm a)].$$

But to conform to the notation of Fig. 1, we must write this as

$$w = re^{\pm i\theta};$$

and upon comparing these equations we readily obtain the relation

$$a = \tan \tfrac{1}{2}\theta. \tag{10}$$

Finally, upon substituting (9) and (10) in (8) we get the result

$$f = \tfrac{1}{2}k \log \frac{\left[\tan^2 \tfrac{1}{2}\theta + \left(\frac{w-r}{w+r}\right)^2\right]^{1/2} - \sec \tfrac{1}{2}\theta}{\left[\tan^2 \tfrac{1}{2}\theta + \left(\frac{w-r}{w+r}\right)^2\right]^{1/2} + \sec \tfrac{1}{2}\theta}, \tag{11}$$

in which, as noted in the footnote accompanying equation (8), the real parts of the radicals are to be taken positive.

This function satisfies the differential equation (1) and the boundary value (2). It also has a logarithmic singularity at the origin. But we are still not sure that the k in (11) has really been so chosen that (3) is satisfied, nor that the function is regular at infinity. To decide these points we must now investigate the behavior of (11) when w is very small and when w is very large.

We readily find that, in the former case

$$f \doteq \tfrac{1}{2}k[\log (w/r) + \pi i + \log \cos^2 \tfrac{1}{2}\theta], \tag{12}$$

and in the latter,

$$f \doteq \tfrac{1}{2}k[-\log (w/r) + \pi i + \log \cos^2 \tfrac{1}{2}\theta]. \tag{13}$$

We first notice that the term $-\frac{1}{2}k \log (w/r)$ in (13) becomes infinite with w. This violates the requirement of regularity at infinity. But as this term is itself a solution of Laplace's equation, which becomes infinite at $w=0$, and the real part of which vanishes over C, both (1) and (2) will be satisfied by $f+\frac{1}{2}k \log(w/r)$ which is regular at infinity, as well as by f itself. Moreover, we easily see from (12) that in the neighborhood of w_0

$$[f + \tfrac{1}{2}k \log (w/r)] \doteq k \log (u^2 + v^2)^{1/2},$$

the remaining terms on the right remaining finite as u and v vanish. Hence (3) also is satisfied by the real part of the function $f+\frac{1}{2}k \log (w/r)$. Thus the solution of our problem is the real part of

$$\phi = \tfrac{1}{2}k \left[\log (w/r) + \log \frac{\left[\tan^2 \tfrac{1}{2}\theta + \left(\frac{w-r}{w+r}\right)^2\right]^{1/2} - \sec \tfrac{1}{2}\theta}{\left[\tan^2 \tfrac{1}{2}\theta + \left(\frac{w-r}{w+r}\right)^2\right]^{1/2} + \sec \tfrac{1}{2}\theta} \right]. \tag{14}$$

This formula (14) is in suitable form for computation when we wish to know the exact potential at any given point (u, v). If, however, we wish to locate an equipotential surface—that is, the locus of all points at which the potential has a given value—it turns out to be more convenient to put (14) in another form. For this purpose, we shall write $s=w/r$ and $t=-\phi/k$, and then solve for s in terms of t. The result is

$$s = \frac{e^{-t} + i \sec \tfrac{1}{2}\theta}{e^{t} - i \sec \tfrac{1}{2}\theta}. \tag{15}$$

If we now assign to t a succession of values all having *the same real part*, and plot the results in the complex s-plane, we will obtain the locus of points at which the potential has the assigned value.

Fig. 4 was produced in this way. It shows the equipotential surfaces about a cylinder, the slit in which is of angular width 20°.

It should be said in explanation of this figure that the heavy circular arc represents the split cylinder, and that the figures attached to the curves are eighteen times[1] the real parts of t to which they correspond. The curves marked 0.01, 0.02, $\cdots$, 0.5, after passing the edge of the slit, lie so close to the inner surface of the cylinder that it is impossible to distinguish between them on the scale of the drawing. These curves, together with those marked 1, 2 and 5, were actually computed from formula (15). The remaining curves were drawn in accordance with the usual logarithmic potential formula,[2] from which they differ so little that it would be impossible to detect the discrepancy on the scale of the drawing.

[1] This apparently arbitrary factor was introduced to afford more convenient comparison with the case where the electrode is a plane from which a slit has been cut. It will be noticed that just 1/18 of our cylinder has been cut away.

[2] That is, the number attached to any circle is eighteen times the negative of the Naperian logarithm of its radius.

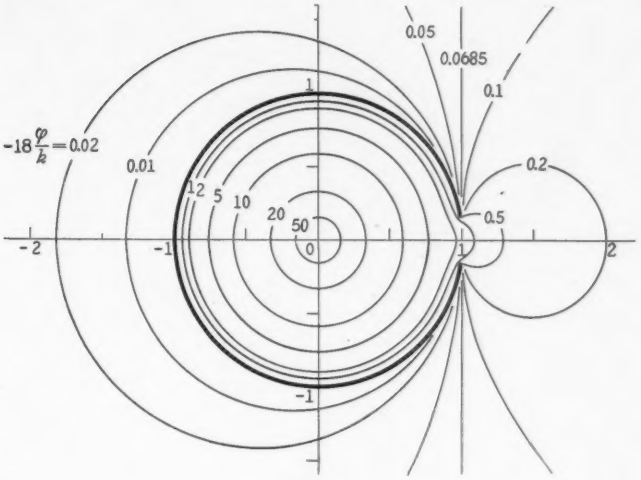


FIG. 4

Two things are immediately evident from this figure. One is that, although our solution has been obtained upon the assumption that the inner conductor is an *infinitely thin* wire, the result actually holds with a very high degree of accuracy, even when the inner cylinder is nearly as large as the outer one. For the field would be unchanged if the inner conductor had the same shape as any equipotential surface; and even so large an oval as that marked 5 differs but little from a circle.

The other is, that the stray field due to the slit in the cylinder is very small, even when the slit is as wide as that shown in the figure.

3. *The Potential Field Between Two Parallel Planes in One of Which There is a Circular Hole.*

The second problem deals with a pair of parallel planes, in one of which there is a circular hole. This problem can also be much simplified by assuming that the planes are infinitely far apart—an assumption which is in all respects analogous to that used in the first problem when we spoke of an "infinitely thin wire."

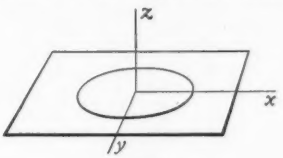


FIG. 5

Our problem may then be stated as that of satisfying Laplace's equation subject to the boundary conditions that: (1) ϕ shall vanish over the entire plane $z=0$ (see Fig. 5), except within a circle of radius r about the origin; (2) that the

gradient $\partial\phi/\partial z$ shall approach a well-defined value k as z becomes positively infinite; and (3) that $\partial\phi/\partial z$ shall approach zero as z becomes negatively infinite.[1]

The essential element in our method of attack upon this problem will be the choice of a suitable coordinate system. Specifically, we shall choose it in such a way that one of the coordinate surfaces occupies exactly that part of the plane $z=0$ which represents the physical electrode.

This can be done most easily by rotating the ellipses and hyperbolas of Fig. 6 about their minor axes, thus generating a set of mutually orthogonal ellipsoids and hyperboloids. In doing this, the hyperboloid $\mu=0$ degenerates into the desired physical electrode; while the ellipsoid $\lambda=0$ degenerates into the area of the hole.

It is easy to see that the curvilinear coordinates λ, μ, θ are related to the Cartesian coordinates x, y, z by the equations

$$
\begin{aligned}
x &= b\cosh\lambda\cos\mu\cos\theta, && 0 \leqq \theta \leqq 2\pi,\\
y &= b\cosh\lambda\cos\mu\sin\theta, && 0 \leqq \mu \leqq \pi/2,\\
z &= b\sinh\lambda\sin\mu, && -\infty \leqq \lambda \leqq \infty,
\end{aligned}
\tag{16}
$$

or what amounts to the same thing,

$$
\begin{aligned}
\tan\theta &= y/x,\\
r^2 &= x^2+y^2+z^2,\\
2b^2\sinh^2\lambda &= r^2-b^2+\left[(r^2-b^2)^2+4b^2z^2\right]^{1/2},\\
2b^2\cosh^2\lambda &= r^2+b^2+\left[(r^2-b^2)^2+4b^2z^2\right]^{1/2},\\
2b^2\sin^2\mu &= b^2-r^2+\left[(r^2-b^2)^2+4b^2z^2\right]^{1/2},\\
2b^2\cos^2\mu &= r^2+b^2-\left[(r^2-b^2)^2+4b^2z^2\right]^{1/2};
\end{aligned}
\tag{17}
$$

b being the radius of the circular hole.

It is not difficult to establish the fact that in this coordinate system the Laplacian operator takes the form[2]

$$
\nabla^2\phi = \frac{1}{b^2(\sinh^2\lambda+\sin^2\mu)\cosh\lambda\cos\mu}\left[\frac{\partial}{\partial\lambda}\cosh\lambda\cos\mu\frac{\partial\phi}{\partial\lambda}\right.
$$

$$
\left.+\frac{\partial}{\partial\mu}\cosh\lambda\cos\mu\frac{\partial\phi}{\partial\mu}+\frac{\partial}{\partial\theta}\frac{\sinh^2\lambda+\sin^2\mu}{\cosh\lambda\cos\mu}\frac{\partial\phi}{\partial\theta}\right].
$$

However, since ϕ is independent of θ by symmetry, all derivatives with respect to θ drop out, and Laplace's equation takes the simpler form

---

[1] This last condition merely asserts that there shall be no third electrode (and hence no charge) *below* the xy-plane in Fig. 5.

[2] See Kellogg, loc. cit., p. 183.

$$\frac{1}{\cosh \lambda}\frac{\partial}{\partial \lambda}\cosh \lambda \frac{\partial \phi}{\partial \lambda} + \frac{1}{\cos \mu}\frac{\partial}{\partial \mu}\cos \mu \frac{\partial \phi}{\partial \mu} = 0. \tag{18}$$

Our problem is therefore to solve this equation subject to the boundary conditions

$$\begin{aligned} \phi &= 0 \quad \text{on} \quad \mu = 0; \\ d\phi/dz &= k \quad \text{at} \quad z = +\infty; \\ d\phi/dz &= 0 \quad \text{at} \quad z = -\infty. \end{aligned} \tag{19}$$

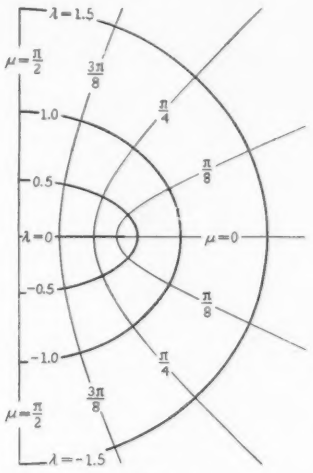


FIG. 6

Upon assuming that ϕ is the product of a function $\Lambda(\lambda)$ which does not depend upon μ and a function $M(\mu)$ which does not depend upon λ, we find that Λ and M must separately satisfy the equations

$$\begin{aligned} \frac{d}{d\lambda}\cosh \lambda \frac{d\Lambda}{d\lambda} - m(m+1)\Lambda \cosh \lambda &= 0, \\ \frac{d}{d\mu}\cos \mu \frac{dM}{d\mu} + m(m+1)M \cos \mu &= 0, \end{aligned} \tag{20}$$

in which m is an arbitrary constant. The last of these is the familiar equation for the surface spherical harmonic[1] of order m; and since the other obviously takes the same form if λ is replaced by $i\lambda$, its solution is a surface harmonic of a pure imaginary argument.

[1] See, for example, Byerly, *Fourier Series and Spherical Harmonics*, pp. 9 and 147. Our μ is related to Byerly's θ by the law $\mu = \frac{1}{2}\pi - \theta$.

We could, therefore, write down a very general solution of (18), building a series in which the terms were all composed of such spherical harmonics, but with various values of m. It turns out, however, that the conditions of our problem can be satisfied by means of just one such term, in which the harmonics are of order 1; so there is no need to use the series at all.[1]

On the other hand, we find by trial that in the case of the function $\Lambda(\lambda)$ we need, not only the well-known "function of the first kind," but also the less familiar "function of the second kind." Our solution, therefore, takes the form

$$\phi = [A \sinh \lambda + B(1 + \sinh \lambda \operatorname{gd} \lambda)] \sin \mu, \tag{21}$$

in which $\operatorname{gd}\lambda$, the "Gudermannian," is identical with $\tan^{-1}\sinh\lambda$, while A and B are constants whose values must be determined so as to satisfy our boundary conditions.[2]

We must now satisfy our boundary conditions (19). That the first is satisfied is at once obvious. To deal with the other two we confine ourselves to the axis $\mu=\pi/2$, along which, by (16), $z=b \sinh \lambda$. Along this line, then,

$$b\phi = Az + B[b + z \tan^{-1}(z/b)],$$

and hence

$$b\partial\phi/\partial z = A + B \tan^{-1}(z/b) + [Bbz/(b^2 + z^2)].$$

The last two of equations (19) now become

$$A + \tfrac{1}{2}\pi B = kb$$

and

$$A - \tfrac{1}{2}\pi B = 0;$$

whence $A=kb/2$ and $B=kb/\pi$. Inserting these values in (21) we get, as the final solution of our problem, subject to all its boundary conditions, the function

---

[1] In other words, if we followed the usual course of writing down a series with arbitrary coefficients, and then determining the coefficients so as to satisfy (19), we would find that all but two vanished. The remaining two would be the A and B of (21).

[2] Owing to the fact that the text-books say so little about spherical harmonics of the second kind, it may be well to prove that $1+\sinh \lambda \operatorname{gd} \lambda$ is really such a function. This can be done, of course, by direct substitution in (20); but it may be of more interest to derive it from first principles.

We know, to begin with, that $\sin \mu$ is the spherical harmonic of the first kind with a real argument, (see Byerly, loc. cit., p. 159), and is a solution of the second of equations (20). But since upon setting $\mu=i\lambda$, the second of equations (20) reduces to the first, it follows that $\sin i\lambda$, and hence $\sinh \lambda$ also, must be a solution of the first.

However, when one such solution is known the complete solution can always be found by variation of parameters. To do this, we suppose that

$$\Lambda(\lambda)=\alpha(\lambda) \sinh \lambda$$

is also a solution, $\alpha(\lambda)$ being as yet undefined. Substituting this in the first of equations (20) we get a solvable equation for $\alpha(\lambda)$, which leads us at once to $1+\sinh \lambda \operatorname{gd} \lambda$ for the function of the second kind with imaginary argument.

$$\phi = \tfrac{1}{2}kb\,[\sinh\lambda + 2\pi^{-1}(1 + \sinh\lambda\,\mathrm{gd}\,\lambda)]\sin\mu. \tag{22}$$

This formula is satisfactory when we wish to find the potential at any given point in space, since we need only find the values of λ and μ which correspond to this point from (17), and then substitute these values in (22). A better form for the plotting of equipotential surfaces, however, is easily obtained by introducing the parameter

$$\beta = \left[\frac{r^2 - b^2 + [(r^2 - b^2)^2 + 4b^2z^2]^{1/2}}{2b^2}\right]^{1/2},$$

which leads to the pair of parametric equations

$$\begin{aligned} z/b &= \frac{\pi\phi/kb}{\tfrac{1}{2}\pi + \tan^{-1}\beta + \beta^{-1}}, \\ r^2/b^2 &= (1 + \beta^2) - z^2b^{-2}\beta^{-2}. \end{aligned} \tag{23}$$

In using these equations to plot the equipotential surface corresponding to a given value of ϕ, the significance of β can be entirely forgotten: it may be assigned any convenient set of values and the coordinates of the corresponding points (z, r) computed.

Fig. 7 shows the equipotential surfaces for this case, the numbers attached to the curves being the values of $\pi\phi/kb$ to which they correspond.

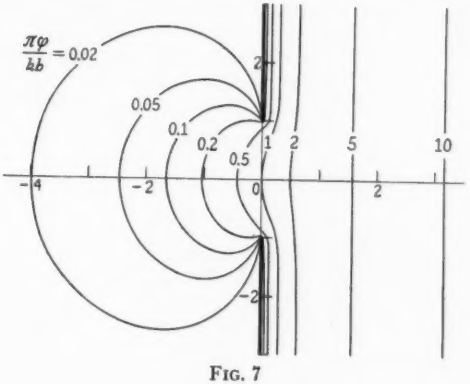


FIG. 7

We are now in a position to justify the use of the formulae (22) and (23) even when the distance between the electrodes is not infinite; for it is obvious that the curves marked 5 and 10 deviate from straight lines by amounts which cannot be detected on the scale of the drawing. Hence we may say, that if the distance between the planes is more than $\frac{3}{4}$ of the diameter of the hole, the solution (22) may be used with a very high degree of accuracy.

THE ROTATION-VECTOR IN ELEMENTARY DIFFERENTIAL GEOMETRY

By REV. M. F. EGAN, University College, Dublin

1. If $\boldsymbol{a}$ and $\boldsymbol{r}$ are vectors, the vector product $\boldsymbol{a}\times\boldsymbol{r}\,du$ represents the displacement of the point whose position-vector is $\boldsymbol{r}$, due to an infinitesimal rotation of amount $\boldsymbol{a}\,du$ about an axis through the origin directed along $\boldsymbol{a}$. We can call $\boldsymbol{a}$, or $\boldsymbol{a}\,du$, the *rotation-vector* for this rotation.

There is nothing new in this idea, but it simplifies a good deal of elementary differential geometry, and gives us some interesting results very easily.

2. Consider a curve in space, with torsion τ and curvature κ. If $\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{b}$ are unit vectors with a fixed origin, and congruent to the unit tangent, principal normal and binormal respectively; then corresponding to the translation ds along the curve, the tripod[1] $(\boldsymbol{tpb})$ has the rotations τds and κds about the tangent and binormal respectively; hence if $\boldsymbol{v}$ is a vector invariably connected with the tripod, we have

$$\frac{d\boldsymbol{v}}{ds} = (\tau\boldsymbol{t} + \kappa\boldsymbol{b}) \times \boldsymbol{v}.$$

Putting $\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{b}$ successively for $\boldsymbol{v}$, we get Frenet's equations.

3. Consider a curve on a surface, and let T be the unit tripod formed by the tangent $(\boldsymbol{t})$, the normal to the curve which is tangential to the surface $(\boldsymbol{s})$, and the normal to the surface $(\boldsymbol{n})$. Let

$$\boldsymbol{a}ds = (\alpha\boldsymbol{t} + \beta\boldsymbol{s} + \gamma\boldsymbol{n})ds$$

be the rotation-vector corresponding to the translation $\boldsymbol{t}\,ds$. If $\boldsymbol{v}$ is a vector invariably attached to T, we have

$$\frac{d\boldsymbol{v}}{ds} = \boldsymbol{a} \times \boldsymbol{v}, \tag{3.1}$$

and putting $\boldsymbol{t}$, $\boldsymbol{s}$, $\boldsymbol{n}$ successively for $\boldsymbol{v}$ we get three equations to determine α, β, γ.

Again, $\boldsymbol{p}$ being as before the principal normal of the curve, if θ is the angle $\boldsymbol{p}\rightarrow\boldsymbol{n}$, the rotation of the tripod T is the same as that of the tripod $(\boldsymbol{tpb})$, along with a rotation $d\theta$ about $\boldsymbol{t}$. We have therefore

$$\begin{aligned}\boldsymbol{a} &= \left(\tau + \frac{d\theta}{ds}\right)\boldsymbol{t} + \kappa\boldsymbol{b} \\ &= \left(\tau + \frac{d\theta}{ds}\right)\boldsymbol{t} - \kappa\boldsymbol{s}\cos\theta + \kappa\boldsymbol{n}\sin\theta,\end{aligned}$$

which gives α, β, γ as respectively the geodesic torsion, minus the normal curvature, and (plus) the geodesic curvature.

[1] "Tripod" seems preferable on all grounds to "trihedron" or "tri-vector." Blaschke (Differentialgeometrie, vol. I, 1930) uses "Dreibein."

4. Let $\alpha', \cdots, \boldsymbol{t}', \cdots, ds'$ refer to one of the parametric curves through a point on a surface S, and $\alpha'', \cdots, \boldsymbol{t}'', \cdots, ds''$ to the other. We denote $\partial/\partial s'$ and $\partial/\partial s''$ by the suffixes 1 and 2. Let ω be the angle $\boldsymbol{t}' \to \boldsymbol{t}''$. If $\boldsymbol{r}$ is the position-vector of the point on S, it is easily seen that

$$\boldsymbol{n}dS = \boldsymbol{r}_1 \times \boldsymbol{r}_2 ds' ds'' = \boldsymbol{n} \sin \omega ds' ds''.$$

Since $\boldsymbol{n}$ is also normal to the unit sphere N (locus of the extremity of $\boldsymbol{n}$), we have in like manner

$$\boldsymbol{n}dN = \boldsymbol{n}_1 \times \boldsymbol{n}_2 ds' ds''.$$

The Gaussian curvature K is dN/dS, we have therefore

$$\begin{aligned} \boldsymbol{n}K \sin \omega &= \boldsymbol{n}_1 \times \boldsymbol{n}_2 \\ &= (\boldsymbol{a}' \times \boldsymbol{n}) \times (\boldsymbol{a}'' \times \boldsymbol{n}) \\ &= (\boldsymbol{a}', \boldsymbol{a}'', \boldsymbol{n})\boldsymbol{n} \end{aligned}$$

whence

$$(4.1) \qquad \begin{cases} K = (\boldsymbol{a}', \boldsymbol{a}'', \boldsymbol{n}) \operatorname{cosec} \omega, \\ \quad = \alpha'\alpha'' + \beta'\beta'' + (\alpha'\beta'' - \alpha''\beta') \cot \omega. \end{cases}$$

We can find J (the sum of the principal normal curvatures) very easily in the same manner. If we consider the parallel surface S', locus of the point

$$\boldsymbol{r}' = \boldsymbol{r} + c\boldsymbol{n},$$

where c is a constant, we have

$$\begin{aligned} \boldsymbol{n}K \sin \omega = \boldsymbol{n}_1 \times \boldsymbol{n}_2 &= K'\boldsymbol{r}_1' \times \boldsymbol{r}_2' \\ &= K'(\boldsymbol{r}_1 + c\boldsymbol{n}_1) \times (\boldsymbol{r}_2 + c\boldsymbol{n}_2). \end{aligned}$$

If we consider the coefficient of c, it is clear that the value of

$$(\boldsymbol{n}_1 \times \boldsymbol{r}_2 + \boldsymbol{r}_1 \times \boldsymbol{n}_2) \operatorname{cosec} \omega,$$

i.e. of

$$\boldsymbol{n}\{\beta' + \beta'' + (\alpha' - \alpha'') \cot \omega\}$$

is independent of the choice of parametric curves. If we take the lines of curvature, the expression in brackets reduces to $\beta' + \beta''$, and since β is minus the normal curvature in the given direction, $\beta' + \beta''$ is $-J$: whence we get in the general case

$$(4.2) \qquad -J = \beta' + \beta'' + (\alpha' - \alpha'') \cot \omega.$$

5. If the element of length on the surface is given by

$$ds^2 = e^2 du^2 + 2eg \cos \omega du dv + g^2 dv^2,$$

the elements of length

$$ds' = e du, \quad ds'' = g dv$$

along the parametric lines are not in general perfect differentials, and so we cannot consider s' and s'' as independent variables. From the identity

$$\int_C Xdu + Ydv = \iint \left(\frac{\partial Y}{\partial u} - \frac{\partial X}{\partial v}\right)dudv$$

we easily deduce the corresponding identity

$$\int_C Pds' + Qds'' = \iint \operatorname{cosec}\omega(Q_1 + q''Q - P_2 - q'P)dS, \tag{5.1}$$

where

$$q' = \frac{1}{eg}\frac{\partial e}{\partial v}, \quad q'' = \frac{1}{eg}\frac{\partial g}{\partial u}. \tag{5.11}$$

(If the parametric lines are orthogonal, we have

$$q' = -\gamma', \quad q'' = \gamma''.)$$

From this we obtain the condition that $P\,ds' + Q\,ds''$ should be a perfect differential. We also have,[1] if

$$\begin{aligned} &\theta_{12} = (\theta_1)_2, \\ &\theta_{12} + q'\theta_1 = \theta_{21} + q''\theta_2. \end{aligned} \tag{5.2}$$

6. If we express that

$$\boldsymbol{t}'ds' + \boldsymbol{t}''ds'' = d\boldsymbol{r}$$

is a perfect differential, we get

$$\boldsymbol{t}_1'' + q''\boldsymbol{t}'' = \boldsymbol{t}_2' + q'\boldsymbol{t}'.$$

Now the rotation of $\boldsymbol{t}''$, corresponding to the displacement $\boldsymbol{t}'\,ds'$, is equal to the corresponding rotation of the tripod T', along with a rotation $\omega_1 ds'$ about $\boldsymbol{n}$. Hence

$$\boldsymbol{t}_1'' = (\boldsymbol{a}' - \boldsymbol{n}\omega_1) \times \boldsymbol{t}'',$$

and in like manner

$$\boldsymbol{t}_2' = (\boldsymbol{a}'' - \boldsymbol{n}\omega_2) \times \boldsymbol{t}'.$$

Substituting these values and remembering that

$$\begin{aligned} \boldsymbol{t}'' &= \boldsymbol{t}'\cos\omega + \boldsymbol{s}'\sin\omega, \\ \boldsymbol{s}'' &= \boldsymbol{n} \times \boldsymbol{t}'' = \boldsymbol{s}'\cos\omega - \boldsymbol{t}'\sin\omega, \end{aligned}$$

we easily get the identities

$$\begin{cases} (\alpha' + \alpha'')\sin\omega = (\beta' - \beta'')\cos\omega, \\ -q' + q''\cos\omega = (\omega_1 + \gamma')\sin\omega, \\ -q'' + q'\cos\omega = (\omega_2 - \gamma'')\sin\omega. \end{cases} \tag{6.1}$$

[1] Cf. Blaschke, *op. cit.* I, p. 125.

The second and third of these identities give a geometrical definition of q' and q''; for it is easy to see that the turning-rates (round $\boldsymbol{n}$) of $\boldsymbol{t}'$ as we go along $\boldsymbol{t}''$, and of $\boldsymbol{t}''$ as we go along $\boldsymbol{t}'$, are respectively $\omega_1+\gamma'$ and $\gamma''-\omega_2$, per unit of length. It follows that the necessary and sufficient conditions that the tangent to each parametric line should undergo a parallel transport (in the Levi-Civita sense) along the other parametric line, are

$$q' = q'' = 0, \tag{6.2}$$

whence

$$\partial e/\partial v = 0, \quad \partial g/\partial u = 0,$$

from which we get at once the well-known theorem that the coordinate lines form a Tchebychef net.

The equation (5.11) shows that q' and q'' are first-order magnitudes, i.e., they depend only on the expression for ds^2. The same is true of ω, and therefore also, by (6.1), of γ' and γ''.

The first of the equations (6.1) shows that the geodesic torsions α' and α'' are equal and of opposite signs when the parametric curves are orthogonal, and also when the normal curvatures are equal. In the case of the asymptotic lines the normal curvatures are both zero, and the geodesic torsions are equal to the actual torsions. We thus get from (4.1) and (4.2) the familiar results about the torsion of an asymptotic line:

$$\tau^2 = -\kappa, \quad 2\tau \cot \omega = -J,$$

where ω is the angle between the asymptotic lines.

7. Let $\boldsymbol{t}, \cdots, T, \cdots, \alpha, ds$ refer to a curve (C); let ϕ be the angle $\boldsymbol{t}' \to \boldsymbol{t}$, and let

$$d\boldsymbol{r} = \boldsymbol{t}ds = \boldsymbol{t}'ds' + \boldsymbol{t}''ds''.$$

Then the rotation of the tripod T corresponding to the translation $d\boldsymbol{r}$ is the sum of (1) that of T' along $\boldsymbol{t}'ds'$, (2) that of T'' along $\boldsymbol{t}''\,ds''$, less a rotation $\omega_2 ds''$ about $\boldsymbol{n}$, and (3) a rotation $d\phi$ about $\boldsymbol{n}$. We therefore have

$$\boldsymbol{a}ds = \boldsymbol{a}'ds' + (\boldsymbol{a}'' - \boldsymbol{n}\omega_2)ds'' + \boldsymbol{n}d\phi. \tag{7.1}$$

If we suppose the coordinate curves *orthogonal*, we have

$$ds' = ds \cos \phi, \quad ds'' = ds \sin \phi,$$

whence we get

$$\left\{\begin{aligned} \alpha &= \alpha' \cos^2 \phi + \alpha'' \sin^2 \phi + (\beta' - \beta'') \sin \phi \cos \phi \\ &= \alpha'(\cos^2 \phi - \sin^2 \phi) + (\beta' - \beta'') \sin \phi \cos \phi, \\ &\qquad \text{since } \alpha' + \alpha'' = 0; \end{aligned}\right. \tag{7.21}$$

$$\beta = (\alpha'' - \alpha') \sin \phi \cos \phi + \beta' \cos^2 \phi + \beta'' \sin^2 \phi, \tag{7.22}$$

$$\gamma = \gamma' \cos \phi + \gamma'' \sin \phi + \frac{d\phi}{ds}. \tag{7.23}$$

For oblique coordinates we have

$$ds/\sin\omega = ds'/\sin(\omega - \phi) = ds''/\sin\phi,$$

and there is no difficulty in writing down the corresponding identities.

These equations are useful in discussing lines of curvature, asymptotic lines, etc.

8. *The Mainardi-Codazzi equations.* If $\boldsymbol{v}'$ is a vector invariably attached to T', we have

$$d\boldsymbol{v}' = \boldsymbol{a}' \times \boldsymbol{v}' ds' + \boldsymbol{b}' \times \boldsymbol{v}' ds'',$$

where

$$\boldsymbol{b}' = \boldsymbol{a}'' - \boldsymbol{n}\omega_2.$$

Hence

$$(\boldsymbol{b}' \times \boldsymbol{v}')_1 + q''\boldsymbol{b}' \times \boldsymbol{v}' - (\boldsymbol{a}' \times \boldsymbol{v}')_2 - q'\boldsymbol{a}' \times \boldsymbol{v}' = 0.$$

Now

$$\boldsymbol{v}_2' = \boldsymbol{b}' \times \boldsymbol{v}', \quad \boldsymbol{v}_1' = \boldsymbol{a}' \times \boldsymbol{v}',$$

and

$$\boldsymbol{a}' \times (\boldsymbol{b}' \times \boldsymbol{v}') - \boldsymbol{b}' \times (\boldsymbol{a}' \times \boldsymbol{v}) = (\boldsymbol{a}' \times \boldsymbol{b}') \times \boldsymbol{v}',$$

so that we get[1]

$$(\boldsymbol{b}_1' + q''\boldsymbol{b}' - \boldsymbol{a}_2' - q'\boldsymbol{a}' - \boldsymbol{a}' \times \boldsymbol{b}') \times \boldsymbol{v}' = 0$$

or, since $\boldsymbol{v}'$ is arbitrary,

$$\boldsymbol{b}_1' + q''\boldsymbol{b}' - \boldsymbol{a}_2' - q'\boldsymbol{a}' = \boldsymbol{a}' \times \boldsymbol{b}'. \tag{8.1}$$

Multiplying *scalarly* by $\boldsymbol{v}'$, and adding to each side

$$\boldsymbol{b}' \cdot \boldsymbol{v}_1' - \boldsymbol{a}' \cdot \boldsymbol{v}_2' = -2(\boldsymbol{a}', \boldsymbol{b}', \boldsymbol{v}'),$$

we get

$$(\boldsymbol{b}' \cdot \boldsymbol{v}')_1 + q''\boldsymbol{b}' \cdot \boldsymbol{v}' - (\boldsymbol{a}' \cdot \boldsymbol{v}')_2 - q'\boldsymbol{a}' \cdot \boldsymbol{v}' = -(\boldsymbol{a}', \boldsymbol{b}, \boldsymbol{v}'). \tag{8.2}$$

If we multiply both sides by cosec ω and integrate over the area bounded by a closed curve (C), we get, in virtue of (5.1),

$$-\iint (\boldsymbol{a}', \boldsymbol{a}'' - \boldsymbol{n}\omega_2, \boldsymbol{v}') \operatorname{cosec}\omega dS = \int_C \boldsymbol{v}' \cdot [\boldsymbol{a}' ds' + (\boldsymbol{a}'' - \boldsymbol{n}\omega_2) ds''] \tag{8.21}$$

$$= \int_C \boldsymbol{v}' \cdot (\boldsymbol{a} ds - \boldsymbol{n} d\phi), \tag{8.22}$$

by (7.1). First, write $\boldsymbol{n}$ for $\boldsymbol{v}'$. We have

$$(\boldsymbol{a}', \boldsymbol{a}'' - \boldsymbol{n}\omega_2, \boldsymbol{n}) = (\boldsymbol{a}', \boldsymbol{a}'', \boldsymbol{n}) = K \sin\omega$$

by (4.1); accordingly from (8.22) we get Gauss' identity

$$-\iint K dS = \int_C \gamma ds - \int d\phi, \quad = \int_C [\gamma' ds' + (\gamma'' - \omega_2) ds''], \tag{8.31}$$

[1] Cf. Campbell, *Differential Geometry*, p. 111.

while from (8.2) we get

$$(8.32) \qquad -K\sin\omega = \gamma_1'' + q''\gamma'' - \gamma_2' - q'\gamma' - \omega_{21} - q'\omega_1.$$

(The right-hand side is symmetrical, since

$$\omega_{21} + q''\omega_2 = \omega_{12} + q'\omega_1.)$$

Putting t' for v' in (8.2), and substituting for ω_1 and ω_2 their values from (6.1), we get

$$(8.33)\ \alpha_1'' \cos\omega - \beta_1'' \sin\omega - \alpha_2' = q'(\alpha' - \alpha'') + (q'\cot\omega - q''\operatorname{cosec}\omega)(\beta' - \beta'').$$

On substituting s', we find in the same way

$$(8.34)\ \alpha_1'' \sin\omega + \beta_1'' \cos\omega - \beta_2' = q'(\beta' - \beta'') - (q'\cot\omega - q''\operatorname{cosec}\omega)(\alpha' - \alpha'').$$

If the coordinate curves are orthogonal, the equations are simplified. The second and third of the Mainardi-Codazzi equations becomes

$$(8.331) \qquad \alpha_2' + \beta_1'' = \gamma'(\alpha' - \alpha'') + \gamma''(\beta' - \beta''),$$

$$(8.341) \qquad \alpha_1'' - \beta_2' = \gamma''(\alpha' - \alpha'') - \gamma'(\beta' - \beta'').$$

$$(\alpha'' = -\alpha').$$

9. As an example of the use of these equations, suppose that the coordinates form a Tchebychef net. We have by (6.2) and (6.1)

$$q' = q'' = 0,\ \gamma' = -\omega_1,\ \gamma'' = \omega_2.$$

Since q' and q'' vanish, we can take s' and s'' as independent variables; and ω_{12} and ω_{21} are identical. Hence (8.31) gives

$$K = -\omega_{12}\operatorname{cosec}\omega$$

and

$$\iint K dS = -\iint \frac{\partial^2\omega}{\partial s'\partial s''} ds'ds''.$$

Integrating this over a quadrilateral formed by two pairs of lines belonging to the net, we find that the excess over 2π of the sum of the angles of the quadrilateral is equal to minus the total curvature of the enclosed area. If K is constant we get Hazzidakis's theorem.[1]

Again, let us look for the surfaces whose asymptotic lines form a Tchebychef net. Taking these lines as coordinate curves, we have

$$\alpha'' = -\alpha',\ \beta' = \beta'' = q' = q'' = 0,$$

whence we get from (8.33–4)

$$\alpha_1' = \alpha_2' = 0,$$

and hence α' is constant, and $K(=-\alpha'^2)$ is also constant. Conversely, if K is

[1] Blaschke, *op. cit.* I, pp. 208–9.

constant, the condition is satisfied. Accordingly Hilbert's theorem, that the asymptotic lines on surfaces of constant curvature form a Tchebychef net, expresses a property peculiar to those surfaces.

10. As a final example let us examine with Bonnet the question of the applicability of two surfaces with correspondence of one or both families of asymptotic lines. If *both* families correspond, the geodesic curvatures γ', γ'' are the same for both surfaces, since they are first-order quantities; the normal curvatures are zero; also we have on each surface

$$\alpha' = -\alpha'' = \pm\sqrt{(-K)},$$

so that the geodesic torsions on one surface are either equal, or equal and opposite, to those on the other. Hence the two surfaces are either congruent or symmetrically equal.

If *one* family on each surface corresponds, take the family and its orthogonal trajectories as parametric lines. Excluding the case of symmetry, we can take

$$\alpha', \alpha'' = -\alpha', \beta' = 0, \gamma', \gamma''$$

to be the same on both surfaces. Let β and B be the values of β'' on the two surfaces. Then the equation (8.341) gives us, for the two surfaces,

$$\begin{aligned}\alpha_1'' &= \gamma''(\alpha' - \alpha'') + \gamma'\beta \\ &= \gamma''(\alpha' - \alpha'') + \gamma' B,\end{aligned}$$

and hence

$$\gamma'(\beta - B) = 0.$$

If $\beta = B$, the two surfaces are congruent; rejecting this, we get $\gamma' = 0$, hence the asymptotic lines which correspond are geodesics and are therefore straight lines. The surfaces are therefore ruled, and the generators correspond.

For ruled surfaces the equation (8.331) is easily integrated.[1] We can write

$$ds^2 = du^2 + g^2 dv^2$$

where

$$g^2 = (u - a)^2 + b^2,$$

a and b being functions of v. The equation in question becomes

$$\alpha_2' + \beta_1'' + \gamma''\beta'' = 0$$

or

$$\frac{1}{g}\frac{\partial \alpha'}{\partial v} + \frac{\partial \beta''}{\partial u} + \frac{1}{g}\frac{\partial g}{\partial u}\beta'' = 0,$$

i.e.

$$\frac{\partial \alpha'}{\partial v} + \frac{\partial}{\partial u}(g\beta'') = 0.$$

If ϕ is the angle which the tangent plane makes with the tangent plane at

[1] This was suggested to me by Mr. P. Gormley.

the central point on the generator, then α', being the rotation-rate of the tangent plane about the generator, is $\partial\phi/\partial u$; and we get

$$\frac{\partial}{\partial u}\left(\frac{\partial\phi}{\partial v} + g\beta''\right) = 0,$$

whence

$$\beta'' = \frac{1}{g}\left(V - \frac{\partial\phi}{\partial v}\right),$$

V being a function of v. Since β and B are both of this form, we can suppose the function V to vary continuously from one expression to the other, so that the deformation can be carried out continuously. Again, equation (4.2) shows us that in this case β'' is $-J$, whence we get

$$J = \frac{1}{g}\left(\frac{\partial\phi}{\partial v} - V\right), \quad \text{and} \quad K = -\alpha'^2 = -\left(\frac{\partial\phi}{\partial u}\right)^2;$$

and it is well known and easily proved geometrically that ϕ is given by

$$\tan\phi = \frac{u-a}{b}, \quad \sin\phi = \frac{u-a}{g}, \quad \cos\phi = \frac{b}{g}.$$

11. Instead of the usual six fundamental magnitudes, namely the three coefficients of ds^2 and L, M, and N, where

$$Ldu^2 + 2Mdudv + Ndv^2 = \boldsymbol{n}\cdot d^2\boldsymbol{r} = -d\boldsymbol{n}\cdot d\boldsymbol{r},$$

we have used e, g, ω (coefficients of ds^2—see §5), and the six components of the two rotation-vectors. These nine are connected by the three relations (6.1), besides the Mainardi-Codazzi equations, so they are equivalent as far as number is concerned to the usual set. Again the rotation-components are easily found in terms of the usual magnitudes; and conversely we have

$$\begin{aligned} -d\boldsymbol{n}\cdot d\boldsymbol{r} &= -\left\{(\boldsymbol{a}'ds' + \boldsymbol{a}''ds'')\times\boldsymbol{n}\right\}\cdot(\boldsymbol{t}'ds' + \boldsymbol{t}''ds'') \\ &= -\beta'ds'^2 + \left\{(\alpha'-\alpha'')\sin\omega - (\beta'+\beta'')\cos\omega\right\}ds'ds'' - \beta''ds''^2, \end{aligned}$$

whence

$$\begin{aligned} L/e^2 &= -\beta', \quad N/g^2 = -\beta'', \\ M/eg &= \tfrac{1}{2}(\alpha'-\alpha'')\sin\omega - \tfrac{1}{2}(\beta'+\beta'')\cos\omega \\ &= \alpha'\sin\omega - \beta'\cos\omega \\ &= -\alpha''\sin\omega - \beta''\cos\omega. \end{aligned}$$

The second and third expressions for M are obtained by means of the first of the equations (6.1). They show that if M vanishes, the component in the tangent plane of the rotation-vector corresponding to a displacement along $\boldsymbol{t}'$ lies along $\boldsymbol{t}''$, and conversely; in other words, that the two directions are conjugate.

12. The method we have used breaks down if we take the null-lines on the surface as coordinate curves. Fortunately, it is particularly easy in this case to get the Mainardi-Codazzi equations by the usual method.

MIXED SYSTEMS OF LINEAR EQUATIONS AND INEQUALITIES

By HELEN M. SCHLAUCH, Hunter College

In papers entitled "Systems of Linear Inequalities," Professor L. L. Dines[1] has found a necessary and sufficient condition for the existence of solutions in systems of linear inequalities, and Professor W. B. Carver[2] has found a different kind of necessary and sufficient condition for the non-existence of such solutions.[3] These theorems naturally suggest a consideration of systems containing both linear equations and linear inequalities. It is the purpose of this paper to give certain necessary and sufficient conditions both for the non-existence and for the existence of solutions of such systems.

Consider the mixed system S, in n unknowns, composed of p equations and $m-p$ inequalities:

$$L_i(x) = 0;\ i = 1, 2, \cdots, p$$

and

$$L_i(x) > 0;\ i = p+1, p+2, \cdots, m$$

where $L_i(x)$ stands for the linear expression:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + b_i.$$

The b's may or may not all be zero.

Since theorems 1 and 2, concerning the non-existence of solutions, may be proved by a method analogous to that used by Professor Carver for pure systems of inequalities, they are merely stated here.

Let M denote the matrix of coefficients, $\|a_{ij}\|$, not including the b's. Then,

THEOREM 1: *If for a system S the rank of the matrix M is m, the system is consistent.*

THEOREM 2: *A necessary and sufficient condition that a given system S be inconsistent is that there should exist a set of $m-p+1$ constants, $k_{p+1}, k_{p+2}, \cdots, k_{m+1}$, such that*

$$\sum_{i=p+1}^{m} k_i L_i(x) + k_{m+1} = 0,$$

for all values of the x's satisfying the p equations; at least one of the k's being positive, and none of them being negative.

Consider again the system S above. It is obvious that if the system S is consistent, a consistent system of inequalities, S', may be derived from it by adding each of the p equations in turn, multiplied by some constant k_i; $i = 1, 2, \cdots, p$ (which may or may not be the same for all the equations) to some one of the $m-p$ inequalities, and by including in this derived system, S', these new in-

[1] Annals of Mathematics; vol. 20 (1919), p. 191.

[2] Annals of Mathematics; vol. 23 (1922), p. 212.

[3] See also two papers by Dines in the Annals of Mathematics; vol. 27 (1925), p. 57, and vol. 28 (1926), p. 41.

equalities and each of the $m-p$ inequalities of S. The inequality to which the multiples of the p equations are added may be chosen quite arbitrarily, and, without loss of generality, may be assumed to be the first. Likewise, the p k's may have any values whatsoever.

Such a derived system S' is of the form:

$$L_{p+1}(x) + k_i L_i(x) > 0;\ i = 1, 2, \cdots, p$$

and

$$L_i(x) > 0;\ i = p+1, p+2, \cdots, m.$$

System S' contains m inequalities all of which must be satisfied by any set of values of the x's which satisfy the original system S.

THEOREM 3: *A necessary and sufficient condition that a mixed system of linear equations and inequalities, such as system S, be satisfied by a set of values $a_1, a_2, \cdots, a_n$, or, briefly, by the point A, is that the derived system of inequalities S' be satisfied by this point A for all values of the k's.*

That the condition is necessary is immediately apparent from the above discussion.

To prove the condition sufficient:

If the system S' is satisfied by the point A for all values of the k's, then the $m-p$ inequalities of system S, since they also appear in S', will be satisfied by this point. It remains to show that the p equations of system S are also satisfied by A.

Suppose one of the equations of S, say the rth equation, were not satisfied by A. Then if this set of values were substituted in it, the resulting sum would be different from zero, that is: $L_r(A) \neq 0$. But since A satisfies the system S',

$$L_{p+1}(A) + k_r L_r(A) > 0 \text{ for all values of } k_r.$$

Or,

$$L_{p+1}(A) > -k_r L_r(A) \text{ for all values of } k_r.$$

But since $L_r(A) \neq 0$, it is possible to choose a value of k sufficiently large in absolute value so that

$$L_{p+1}(A) < -k_r L_r(A).$$

Therefore the hypothesis was false, and $L_r(A) = 0$. The same argument applies to each equation of system S, and hence the entire system S is satisfied by the point A.

DEFINITION: A system of p equations and $m-p$ inequalities will be called a *simple* system when the rank of the augmented matrix of the coefficients of the p equations is p.

THEOREM 4: *From the simple system S, of p equations and m-p inequalities, 2^p systems of inequalities may be derived by including the $m-p$ inequalities of system S and one of the 2^p possible combinations of the p inequalities found by re-*

placing the equality symbol $(=)$ *in each equation by one of the inequality symbols* $(>$ *or* $<)$. *A necessary and sufficient condition that the simple system* S *be consistent is that each derived system of inequalities be consistent.*

To prove the condition sufficient:

We are given that there exist 2^p points, $P_1, P_2, \cdots, P_{2^p}$, each giving positive signs for the last $m-p$ expressions of system S and each giving one of the possible combinations of plus and minus signs for the first p expressions when they are substituted in the expressions of system S. Half of these points will make the pth expression positive and half will make it negative. Both of these types will appear with every possible combination of signs of the remaining $p-1$ first expressions. We may then pair our points in groups of two; P_i will be paired with P_j if P_i makes the pth expression positive, P_j makes the pth expression negative, and they both result in the same combination of signs for the first $p-1$ expressions. There will be 2^{p-1} such pairs.

For example:

Some P_i gives the signs:

$$- - - \cdots - -, + + + \cdots +$$

a corresponding P_j gives the signs:

$$- - - \cdots - +, + + + \cdots + .$$

Let us define a point F between C and D as the set of numbers

$$(h_1c_1 + h_2d_1,\ h_1c_2 + h_2d_2,\ \cdots,\ h_1c_n + h_2d_n)$$

where h_1 and h_2 are any two positive numbers whose sum is unity. Since the expressions of system S are linear, we have:

$$L_i(F) = h_1L_i(C) + h_2L_i(D);\ i = 1, 2, \cdots, m .$$

It is then possible for each pair P_i and P_j above, to find a point Q_k; $k=1, 2, \cdots, 2^{p-1}$ making the pth expression $=0$ and the last $m-p$ expressions positive. The signs of the first $p-1$ expressions will be the same as those determined by P_i and P_j. We have, then, 2^{p-1} points making the last $m-p$ expressions >0, the pth expression $=0$, and the first $p-1$ have all possible combinations of the inequality symbols, $(>$ or $<)$.

If we continue this process p times, we are lead to a point R which, when substituted in system S gives:

$$0\ 0\ 0\ 0 \cdots 0, + + + \cdots + .$$

That is, point R satisfies system S, and system S is consistent.

To prove the condition necessary:

We are given that system S is consistent. That is, we are given that there exists a point, Q, which satisfies all the conditions of system S. We wish to find 2^p points which will satisfy the 2^p derived systems described in the theorem.

Let us here define $L_i'(x)$ to be $L_i(x)-b_i$.

Now, consider the possible points of the form: $(q_1+y_1, q_2+y_2, \cdots, q_n+y_n)$ where the y's may have any values whatever.

When a point of this form is substituted in the linear expressions $L_i(x)$, it will give:

$$(A) \qquad L_i'(Y); \qquad i = 1, 2, \cdots, p$$

and

$$L_i(Q) + L_i'(Y); \quad i = p+1, p+2, \cdots, m.$$

If it is possible to find 2^p points of the form $(Q+Y)$ such that when they are substituted in the first p expressions of (A) they give, respectively, the 2^p sets of inequality symbols ($>$ or $<$) required by the theorem, and such that each of them gives

$$|L_i'(Y)| < L_i(Q); \; i = p+1, p+2, \cdots, m,$$

then each of the 2^p derived systems of inequalities described in the theorem will be consistent.

Now consider the set (B) of p non-homogeneous equations obtained by setting

$$(B) \qquad L_i'(x) = R_i; \; i = 1, 2, \cdots, p,$$

where each R_i is some constant.

The rank of the augmented matrix of coefficients of the p equations in system S is p; and since the complete system S is consistent, the rank of the unaugmented matrix of coefficients of the equations (i.e., not including the b's) must also be p.[1] But the unaugmented matrix of the coefficients of (B) is exactly this unaugmented matrix of the coefficients of the p equations of system S. There is therefore at least one point Y satisfying the equations of (B) for any values of the R_i's; $i=1, 2, \cdots, p$.

Then to obtain a point Y which will satisfy any one of the 2^p derived systems of the inequalities, set $R_i = \pm 1$; $i=1, 2, \cdots, p$, according as the ith expression in the particular derived system under consideration is to be $>$ or <0. There will exist a point M satisfying the equations of (B) with these values of the R's. Now choose a k such that $k>0$ and $k > |L_i'(M)|/L_i(Q)$; $i=p+1, p+2, \cdots, m$. Then also, $|L_i'(M)|/k < L_i(Q)$; $i=p+1, p+2, \cdots, m$. Divide the set of values composing the point M by k, giving a point Y of the type described above. This particular point Y is the desired point for, since $L_i'(X)$ is a homogeneous linear expression,

$$|L_i'(Y)| = |L_i'(M)|/k < L_i(Q); \; i = p+1, p+2, \cdots, m,$$

and $L_i(Y) = \pm 1/k$; $i=1, 2, \cdots, p$ according as the ith expression in the derived system under consideration is >0 or <0.

We can then find 2^p points $(Q+Y_i)$; $i=1, 2, \cdots, 2^p$, satisfying the 2^p particular derived systems required by the theorem.

[1] Bôcher: Introduction to Higher Algebra, p. 46.

It may at first appear that the restriction of the theorem to *simple* systems might limit its application. However, even if we have a system S of p equations and $m-p$ inequalities which is not simple, theorem (4) will give a complete test. For, if we first examine the p equations and find that they are inconsistent, we do not need to examine system S further, for it is impossible for the complete system to be consistent if the equations are not themselves consistent. If, however, the p equations are consistent but have a matrix of coefficients whose rank is $r<p$, then any solution of the r equations whose coefficients enter into a non-vanishing r-rowed determinant of the matrix will be a solution of all p equations.[1] We may test the simple system composed of these r equations and the original $m-p$ inequalities by theorem (4). The results will then apply to the non-simple system S.

ALGEBRAIC CHARTS

By EDGAR DEHN, Columbia University

To find simple devices for doing long computations, this task has always held a peculiar charm; and mathematicians have often invented such devices before there was an urgent demand for them, thus anticipating the needs of advancing science.

Tools like logarithms and the slide rule have long since become indispensable in the outfit of every scientific worker. And now the demand is growing for those devices which nomography has to offer. Among such devices are charts for solving algebraic equations, both in one and several unknowns. In this article I confine myself to equations in one unknown, explaining two principles for constructing charts which I found accidentally over a year ago. That anything at all could be added to a subject so exhaustively treated by d'Ocagne merely shows that much remains to be done in this field.

1. The first principle applies to trinomial equations of the form

$$x^n + px + q = 0$$

with real roots, and consists in plotting the coefficients p and q as functions of two roots.

In case of the quadratic equation

$$x^2 - px + q = 0,$$

we have only two roots, x_1 and x_2, with the relations

$$x_1 + x_2 = p, \quad x_1x_2 = q.$$

The first one gives straight lines for parametric values of p, the other equilateral hyperbolas for parametric values of q, if we plot in the x_1x_2-plane. The coördi-

[1] Bôcher, *loc. cit.*, p. 45.

nates of any point where the proper line p and curve q intersect are the roots of a given equation.

If we assume that p, q, x_1, x_2 are positive, we can put every quadratic equation either into the form just used, or else into the form

$$x^2 - px - q = 0$$

with the relations

$$x_1 - x_2 = p, \quad x_1x_2 = q$$

between the roots. These relations give the same hyperbolas but other lines; and plotting both pairs of relations in the first quadrant, we can solve there any given equation. We do not even need the whole quadrant: since we have symmetry about the 45-degree line through the origin, one half of the first quadrant suffices.

In case of the cubic equation

$$x^3 - px + q = 0,$$

we have the relations

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1x_2 + x_1x_3 + x_2x_3 &= -p \\ x_1x_2x_3 &= -q \end{aligned}$$

between the roots. Eliminating x_3, we obtain from these

$$\begin{aligned} x_1^2 + x_1x_2 + x_2^2 &= p \\ x_1^2x_2 + x_1x_2^2 &= q. \end{aligned}$$

The first relation gives ellipses for parametric values of p, and the other gives, for parametric values of q, cubic curves which consist of three branches each. Whenever the roots of a given equation are different, the proper curves p and q intersect in six points. The coördinates of any such point give two roots, of course not always the same roots, and the third root is minus the sum of the other two. The plane of the points thus is the plane of two roots, but not of any definite two roots; we might call it the (x_i, x_j)-plane.

Since any trinomial cubic can be changed so that two of its roots are positive, we need again the curves p and q in the first quadrant alone; and even one half of the quadrant will suffice on account of the same symmetry about the 45-degree line through the origin.

In case of the biquadratic equation

$$x^4 - px + q = 0,$$

we use

$$\sum x_i = \sum x_ix_j = 0$$

to eliminate x_3 and x_4, thus obtaining the relations

$$x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 = p$$
$$x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 = q.$$

Again we can plot in the plane of two indefinite roots, but a chart so constructed would be of little practical value since ridding a biquadratic equation of its terms in x^3 and x^2 is more difficult than solving it by other available means. And the corresponding difficulty for the quintic equation is increased.

Charts constructed on the principle explained seem to give results a great deal more accurate than those obtained by any of the previously known graphic methods. Even a small sized but well drawn chart could be made to yield four significant figures.

2. The second principle for constructing algebraic charts is partly anticipated by Lalanne.

Any equation

$$x^n + px + q = 0$$

is for a definite value of x a line in the pq-plane. Drawing such lines for parametric values of x, one obtains the chart of Lalanne. To solve a given equation, one locates the point (p, q) on the chart and finds or interpolates those lines which go through the point. The x's of those lines are the roots of the equation.

In case of the quadratic equation, two lines go through a point corresponding to an equation with real roots; in the case of the cubic equation, as many as three lines. This makes it difficult to locate the lines, and near their intersections it is almost impossible to do so. I therefore left out the lines entirely in my charts and kept only their envelopes given by

$$\begin{cases} x^n + px + q = 0 \\ nx^{n-1} + p = 0, \end{cases}$$

marking on the envelope each value of x where the line for that value touches the envelope. Numerically such a value determines the slope of the line and the envelope.

The roots of a given equation then are found by holding a ruler so that it passes through the point (p, q) and touches the envelope, the number marked at any contact point being a root. A point (p, q) located on the envelope itself gives a double root.

The envelope for the quadratic equation

$$x^2 + px + q = 0$$

is

$$p^2 = 4q,$$

and the envelope for the cubic equation

$$x^3 + px + q = 0$$

is

$$4p^3 + 27q^2 = 0.$$

While such charts are quickly drawn and very clear, every trinomial equation requiring just one curve, with regard to accuracy they are much inferior to the charts discussed in section 1: only two significant figures can be obtained on the average.

On the other hand, the principle may be used to construct charts for equations with four terms, for instance

$$x^n + px^2 + qx + r = 0.$$

Taking one literal coefficient as a parameter, we have

$$\begin{cases} x^n + px^2 + qx + r = 0 \\ nx^{n-1} + 2px + q = 0 \end{cases}$$

to determine envelopes in the plane of the other two literal coefficients, and all such envelopes compose a chart in that plane. To find the roots of a given equation, one holds a ruler passing through the point of two coefficients and touching the envelope of the third. Any contact point gives a root as before, and the slope at the point is either the root itself or a power of the root.

Points of different envelopes which carry the same number and have the same slope are all on a straight line, say slope line; and drawing the slope lines for different numbers is the best way of marking the numbers.

The cubic equation

$$x^3 + px^2 + qx + r = 0$$

with p as a parameter has the envelopes

$$\begin{cases} q = -2px - 3x^2 \\ r = px^2 + 2x^3 \end{cases}$$

and the slope lines

$$qx + 2r = x^3.$$

Each envelope has a cusp, and the locus of cusps is the envelope

$$\begin{cases} q = 3x^2 \\ r = -x^3 \end{cases}$$

of the slope lines.

A point (q, r) on the envelope p gives in general a double root, but it gives a triple root on the locus of cusps.

The biquadratic equation

$$x^4 + px^2 + qx + r = 0$$

with q as a parameter has the envelopes

$$\begin{cases} p = -2x^2 - q/2x \\ r = x^4 - qx/2 \end{cases}$$

and the slope lines

$$px^2 - r + 3x^4 = 0.$$

Each envelope has a cusp, and the locus of cusps is the envelope

$$p^2 + 12r = 0$$

of the slope lines. A point (p, r) on the envelope q gives in general a double root, but two double roots differing in their sign on the envelope $q=0$ and a triple root on the locus of cusps.

Although the accuracy obtainable on the last charts does not go beyond two significant figures, even solutions with deficient accuracy should be useful if only for further approximation. And the charts give a clear insight into the way the roots change when the coefficients of an equation are being altered.

ON THE PERPENDICULARS FROM A POINT ON A CIRCLE TO THE SIDES OF A REGULAR CIRCUMSCRIBED N-GON

By H. GROSSMAN, De Witt Clinton High School, New York

THEOREM: *If a is the radius of the circle inscribed in a regular n-gon and p_i the length of the perpendicular from any point of the circle on the i-th side of the n-gon, then*

$$\sum_{i=1}^{n} \left(\frac{p_i}{a}\right)^r = n \sum_{j=0}^{[r/2]} \frac{1}{2^{2j}} \binom{r}{2j} \binom{2j}{j} \quad [r = 1, 2, 3, \cdots, n-1].$$

PROOF: We shall need first the following lemma:
If m, n, and k are integers, $n > 1$, $m \not\equiv 0 \bmod n$, and ϕ is any real number, then

$$\sum_{k=0}^{n-1} \cos\left(\phi + \frac{mk}{n} 2\pi\right) = 0.$$

For, if z is a primitive nth root of unity, and w the complex number whose absolute value is 1 and whose amplitude is ϕ, and $R(u)$ indicates the real part of u,

$$\sum_{k=0}^{n-1} \cos\left(\phi + \frac{mk}{n}2\pi\right) = R\left(\sum_{k=0}^{n-1} wz^{mk}\right)$$

$$= R\left[w\left(\frac{1 - z^{mn}}{1 - z^m}\right)\right]$$

$$= R(0)$$

$$= 0,$$

since $z^{mn} = 1 \neq z^m$.

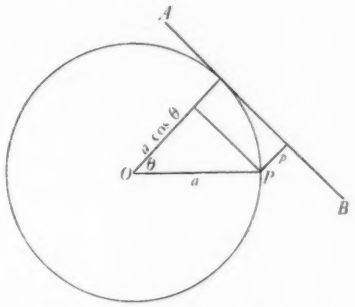
We shall employ also the trigonometric identities:[1]

$$2^{t-1}\cos^t\theta = \cos t\theta + \binom{t}{1}\cos(t-2)\theta + \binom{t}{2}\cos(t-4)\theta + \cdots$$

$$+ \binom{t}{\frac{t-1}{2}}\cos\theta \text{ if } t \text{ is odd;}$$

$$2^{t-1}\cos^t\theta = \cos t\theta + \binom{t}{1}\cos(t-2)\theta + \binom{t}{2}\cos(t-4)\theta + \cdots$$

$$+ \tfrac{1}{2}\binom{t}{\frac{t}{2}} \text{ if } t \text{ is even.}$$



Let O be the center of the circle, P be the point on the circle from which the n perpendiculars are dropped, and AB be that side of the circumscribed regular n-gon the perpendicular to which from O makes the smallest non-negative angle θ with OP.

[1] A Treatise on Plane and Spherical Trigonometry, J. B. Chauvenet, p. 142, §245.

Then

$$\Sigma\left(\frac{p_i}{a}\right)^r = \sum_{k=0}^{n-1}\left[1-\cos\left(\theta+\frac{k}{n}2\pi\right)\right]^r$$

$$= \Sigma 1 - \binom{r}{1}\Sigma\cos\left(\theta+\frac{k}{n}2\pi\right) + \binom{r}{2}\Sigma\cos^2\left(\theta+\frac{k}{n}2\pi\right) - \cdots$$

$$+ (-1)^{r-1}\binom{r}{1}\Sigma\cos^{r-1}\left(\theta+\frac{k}{n}2\pi\right) + (-1)^r\Sigma\cos^r\left(\theta+\frac{k}{n}2\pi\right)$$

$$= \Sigma 1 - \binom{r}{1}\Sigma\cos\left(\theta+\frac{k}{n}2\pi\right) + \binom{r}{2}\Sigma\tfrac{1}{2}\left[\cos\left(2\theta+\frac{2k}{n}2\pi\right)+\tfrac{1}{2}\binom{2}{1}\right]$$

$$- \binom{r}{3}\Sigma\frac{1}{2^2}\left[\cos\left(3\theta+\frac{3k}{n}2\pi\right)+3\cos\left(\theta+\frac{k}{n}2\pi\right)\right]$$

$$+ \binom{r}{4}\Sigma\frac{1}{2^3}\left[\cos\left(4\theta+\frac{4k}{n}2\pi\right)+4\cos\left(2\theta+\frac{2k}{n}2\pi\right)+\tfrac{1}{2}\binom{4}{2}\right]$$

$$- \cdots$$

$$= \Sigma 1 + \binom{r}{2}\Sigma\tfrac{1}{2}\cdot\tfrac{1}{2}\binom{2}{1} + \binom{r}{4}\Sigma\frac{1}{2^3}\cdot\frac{1}{2}\binom{4}{2} + \cdots$$

$$= \sum_{k=0}^{n-1}\left[1+\binom{r}{2}\frac{1}{2^2}\binom{2}{1}+\binom{r}{4}\frac{1}{2^4}\binom{4}{2}+\cdots\right]$$

$$= n\sum_{j=0}^{[r/2]}\frac{1}{2^{2j}}\binom{r}{2j}\binom{2j}{j} \qquad (r<n).$$

For this result, the restriction in the lemma that m is not a multiple of n makes it necessary in the last three steps of the proof that $r<n$ and so limits to $n-1$ the number of conditions to which the theorem subjects the n perpendiculars.

If we remove this limitation on r however and let $r=n$, the term

$$(-1)^n\Sigma\cos^n\left(\theta+\frac{k}{n}2\pi\right)$$

would give, after the trigonometric transformation, one non-vanishing term involving θ:

$$(-1)^n\Sigma\frac{1}{2^{n-1}}\cos(n\theta+k\cdot 2\pi) = \frac{(-1)^n n\cos n\theta}{2^{n-1}},$$

so that we may add the nth condition:

$$\Sigma\left(\frac{p_i}{a}\right)^n = n\left[\sum_{j=0}^{[n/2]}\frac{1}{2^{2j}}\binom{n}{2j}\binom{2j}{j}+\frac{(-1)^n\cos n\theta}{2^{n-1}}\right].$$

With appropriate changes the theorem may be extended to the case of a regular inscribed n-gon. In this case

$$\sum_{i=1}^{n}\left(\frac{p_i}{a}\right)^r$$

assumes the form

$$n\sum_{j=0}^{[r/2]}\frac{1}{2^{2j}}\binom{r}{2j}\binom{2j}{j}\left(\cos\frac{\pi}{n}\right)^{r-2j} \qquad [r=1,2,3,\cdots,n-1].$$

It is necessary however, to make the following important reservation: for that one of the sides of the n-gon on whose minor arc the point P lies, the perpendicular p must be taken negative.

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE SOLUTION OF CUBIC EQUATIONS

By W. C. RISSELMAN, Arizona State Teachers College at Flagstaff

The purpose of this note is to give in symmetric form suitable for computational purposes the solution of cubic equations. Circular and hyperbolic functions will be used. Among the writers who have used hyperbolic functions in the solution of cubics are Grunert,[1] who obtained the results in this paper but did not use the fact that the period of the hyperbolic sine and the hyperbolic cosine is $2\pi i$; and Gleason,[2] who determined only the real root by hyperbolic functions and then gave a trigonometric determination of the other roots. Dickson[3] gives Case 1 below.

Consider the equation

$$y^3+py+q=0.$$

Let $D=-4p^3-27q^2$ denote its discriminant. Let $y=nz$. Then

$$z^3+(p/n^2)z+q/n^3=0. \tag{1}$$

Case 1. $D>0$.

Comparison of (1) with the identity

$$\cos^3x-\tfrac{3}{4}\cos x-\tfrac{1}{4}\cos 3x=0$$

gives

$$n=(-4p/3)^{1/2},\quad \cos 3x=\frac{-q/2}{(-p^3/27)^{1/2}}.$$

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[1] Archiv der Mathematik und Physik, vol. 38, (1862), pp. 48–76.

[2] Annals of Mathematics, ser. 2, vol. 13, pp. 120–122.

[3] First Course in the Theory of Equations, p. 49.

The roots are

$$y_1 = (-4p/3)^{1/2} \cos x,$$
$$y_2 = (-4p/3)^{1/2} \cos (x + 2\pi/3),$$
$$y_3 = (-4p/3)^{1/2} \cos (x + 4\pi/3).$$

Case 2. $D<0$, $p>0$.

Comparison of (1) with the identity

$$\sinh^3 x + \tfrac{3}{4} \sinh x - \tfrac{1}{4} \sinh 3x = 0$$

gives

$$n = (4p/3)^{1/2}, \quad \sinh 3x = \frac{-q/2}{(p^3/27)^{1/2}}.$$

The roots are

$$y_1 = (4p/3)^{1/2} \sinh x,$$
$$y_2 = (4p/3)^{1/2} \sinh (x + 2\pi i/3) = (4p/3)^{1/2}(-\sinh x + i\sqrt{3} \cosh x)/2,$$
$$y_3 = (4p/3)^{1/2} \sinh (x + 4\pi i/3) = (4p/3)^{1/2}(-\sinh x - i\sqrt{3} \cosh x)/2.$$

Case 3. $D<0$, $p<0$, $q<0$.

Comparison of (1) with the identity

$$\cosh^3 x - \tfrac{3}{4} \cosh x - \tfrac{1}{4} \cosh 3x = 0$$

gives

$$n = (-4p/3)^{1/2}, \quad \cosh 3x = \frac{-q/2}{(-p^3/27)^{1/2}}.$$

The roots are

$$y_1 = (-4p/3)^{1/2} \cosh x,$$
$$y_2 = (-4p/3)^{1/2} \cosh (x + 2\pi i/3) = (-4p/3)^{1/2}(-\cosh x + i\sqrt{3} \sinh x)/2,$$
$$y_3 = (-4p/3)^{1/2} \cosh (x + 4\pi i/3) = (-4p/3)^{1/2}(-\cosh x - i\sqrt{3} \sinh x)/2.$$

The case in which $D<0$, $p<0$, and $q>0$ may be brought under Case 3 since changing the sign of q merely changes the signs of the roots. The case in which $D=0$ is trivial.

A Trisection

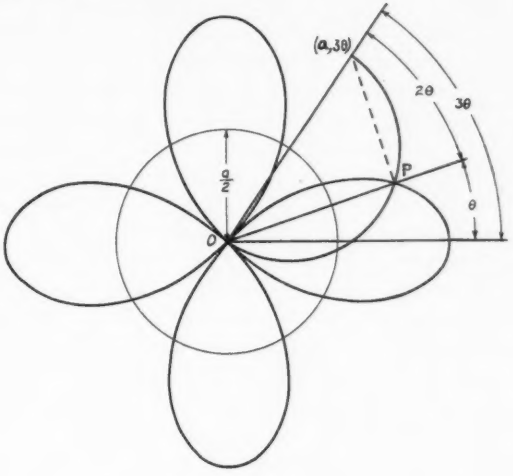
By R. K. Morley, Worcester Polytechnic Institute

That the four-leaved rose, $\rho = a \cos 2\theta$, is a very neat trisectrix does not seem to be generally known.

If a point $(a, 3\theta)$ be projected on the terminal side of θ at P, it is clear that the locus of P as θ changes is the curve $\rho = a \cos 2\theta$. Consequently if the given angle to be trisected is 3θ, a circle with center at $(a/2, 3\theta)$ will cut the rose on the terminal side of θ, the required one-third.

The circle in the figure with center at O and radius $a/2$ is convenient for locating the center $(a/2, 3\theta)$ for different values of 3θ.

Extension of the method to other sections is obvious; e.g. to finding fifth parts of angles by means of $\rho = a \cos 4\theta$.



RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

W. R. Hamilton. Mathematical Papers. Vol. 1. Geometrical Optics. Cambridge, University Press, 1931. xxviii+534 pages.

As to the general plan of publishing Hamilton's Works we cannot improve the statement of the editors in their preface:—"This collected edition of the Mathematical Papers of Sir William Rowan Hamilton has been undertaken by the Royal Irish Academy with three main objects; first, as a tribute to the memory of Hamilton as a distinguished member and President of the Academy, and the greatest of Irish mathematicians; secondly to satisfy a need long felt by those to whom the published papers were difficult of access; and thirdly to publish for the first time those parts of the manuscript which have an historical or modern importance."

The present volume contains Hamilton's optical papers, beginning with his first published paper, *Theory of Systems of Rays. Part First*, presented to the Academy in 1827. In all there are fourteen previously published papers, the last

being dated 1841. They occupy 343 pages. In addition there are seven papers taken from his voluminous manuscripts and now published for the first time. These take up 115 pages. Besides these the volume contains:—The *Eloge on Hamilton* by his life long friend, the Very Rev. Charles Graves, read at the Academy directly after Hamilton's death; a photograph of Hamilton taken about 1845; also a photographic reproduction of a page of manuscript.

A notable feature of the present volume is the large number of notes added by the editors, Professors A. W. Conway and J. L. Synge, both of the University of Dublin. The editors have measured their duties by the severest standards; apparently they have read their way through this stupendous body of mathematical formulae, line by line. Errors have been noted and set right, and when the line of proof is obscure they have not only added necessary foot notes but have also appended a series of twenty-seven notes filling fifty-five pages. These notes are of great value; they are partly historical, partly explanatory, developing more fully the main text. In their last note, they have taken the trouble of applying Hamilton's ideas to compute the coefficients of aberration for an instrument of revolution. The reviewer has a large experience with Collected Works, be they English, German or French; he feels it a pleasant duty to state that he cannot recall any which have been edited with more meticulous care and felicity than the present volume.

Hamilton was born in 1805 and died in 1865. Already as a child he showed precocious ability. At seven he was studying Hebrew, at twelve he had a considerable knowledge of Sanskrit, Hindustani, Persian, and Arabic, as well as the classics and modern European languages. At sixteen he had mastered a great part of Newton's *Principia*, at seventeen he was studying Laplace's *Mécanique Céleste*. At eighteen he entered Trinity College, Dublin. The next year he presented his first paper "On Caustics" to the Royal Irish Academy, (1824). The Committee to whom it had been referred reported: "that the discussions included in the Memoir are of a nature so very abstract and the formulae so general as to require that the reasonings by which some of the conclusions have been obtained should be more fully developed." This paper found among Hamilton's manuscripts is now published for the first time. Remodelled and vastly extended, it was planned to form an immense memoir entitled *Theory of Systems of Rays. Part I, An ordinary System of reflected Rays. Part II, An ordinary System of refracted Rays. Part III, On extraordinary Systems, and Systems of Rays in general.* Part I was published in the Transactions of the Academy 1828. Part II is now published for the first time in the present volume. Part III has never appeared; it was perhaps never written. Instead Hamilton published three *Supplements* 1830–33.

These memoirs stamped Hamilton as one of the great mathematicians of his day. Brinkley, astronomer royal for Ireland early recognized his extraordinary abilities, for already in 1823 he is credited as saying "This young man, I do not say *will be*, but *is* the first mathematician of his age." This must have been a

pretty general opinion, for in 1827 while still an undergraduate he was made Andrews professor of astronomy in the university of Dublin.

Although Hamilton resided at the observatory Dunsink near Dublin for the rest of his life, it was not expected of him to become a great observer. Indeed the university had scant funds to replace the obsolete instruments by ones of modern date. They wished to assure him an honorable position which would permit him to develop his talents to their full extent. In this they amply succeeded.

In 1835 Hamilton was knighted, in 1837 he was elected president of the Dublin Academy. Already in 1827 Hamilton had noted the extension of his optical ideas to dynamics. His first papers on this matter appeared in the Transactions of the Royal Society of London 1834–35. In 1844 Hamilton discovered quaternions, and the last years of his life were chiefly spent in developing this new field.

Let us turn now for a moment to what is peculiar to Hamilton's treatment of optical problems. If we have given an optical instrument consisting of a number of lenses of known curvature and indices of refraction, placed in a certain position, we can by trigonometry trace the path of a ray through the instrument. By varying the ray, we find the rays issuing from a point do not in general meet at a point. Various defects in the image arise such as spherical aberration, coma, curvature of the field etc. The designer of an optical instrument endeavors to remove these various imperfections as far as possible by varying the number of the lenses, their curvature and position. The difficulties of this problem are immense, and it is of course of superlative importance to have certain general principles to guide the practical optician in his work of trial and error. Among the older authors writing from this point of view we may mention Airy and Coddington in England, Gauss and Seidel in Germany.

The point of view of Hamilton is quite different from that of these writers. He asks what one can say of any optical instrument quite disregarding its component parts. His methods seem to be more general and more flexible than any other known today.

Let a ray meet two planes in the points $P_0(x_0y_0z_0)$, $P_1(x_1y_1z_1)$ in an optical system formed of media of indices of refraction $\mu_1, \mu_2, \mu_3, \cdots$, the lengths of the paths in these media being $l_1, l_2, l_3, \cdots$. Then $V=\Sigma\mu l$, or more generally $\int \mu ds$ is called the optical length of the ray joining P_0 and P_1. It is a function of the coordinates $x_0, y_0, z_0, x_1, y_1, z_1$ of the terminal points. Then $V=V(x_0, y_0, z_0, x_1, y_1, z_1)$ is a characteristic function. If we keep P_0 fixed and vary P so that $V=$constant, we get the wave surface of light issuing from P_0. For from Fermat's principle the path of a ray of light joining P_0, P_1 is such that $\delta V=0$. From this we can infer that $l_0, m_0, n_0; l_1, m_1, n_1$ being the direction cosines of the ray at P_0, P_1 are given by

$$\frac{\partial V}{\partial x_0} = l_0, \quad \frac{\partial V}{\partial y_0} = m_0, \quad \frac{\partial V}{\partial z_0} = n_0,$$

and similar relations at P_1.

Instead of $x_0y_0z_0$; $x_1y_1z_1$ we may introduce other variables. For example Hamilton considers

$$T = l_1x_1 + m_1y_1 + n_1z_1 - l_0x_0 - m_0y_0 - n_0z_0 - V$$

regarding the end points P_0, P_1 fixed and the directions of the rays at these points as variable. As $l_0^2+m_0^2+n_0^2=1$, $l_1^2+m_1^2+n_1^2=1$, T is a function of four variables. For any instrument symmetric about an axis, T is a function of three variables

$$u_1=l_0^2+m_0^2, \quad u_1=l_1^2+m_1^2, \quad u_3=l_0l_1+m_0m_1.$$

If we develop T as a function of these variables, supposed small, and neglect small quantities of order higher than the second we get

$$T = T_0 + T_1 + T_2$$

where T_0 is a constant $T_1=a_1u_1+a_2u_2+a_3u_3$ and

$$T_2 = a_{11}u_1^2 + a_{22}u_2^2 + a_{33}u_3^2 + 2a_{12}u_1u_2 + 2a_{13}u_1u_3 + 2a_{23}u_2u_3 .$$

A third choice of variables is to suppose the initial position and final direction variable, leading Hamilton to a third characteristic function $W=l_1x_0+m_1y_0+n_1z_0-V$. These functions form the foundation of Hamilton's optical investigations and by their aid Hamilton discovered many of the results of later workers, for example those relating to the celebrated formulae of Seidel and Petzval.

Hamilton's optical methods lay unused and all but forgotten until quite recently. Thomson and Tait (1867), Maxwell (1875), M. Thiesen (1890), H. Bruns (1895) are the only writers the reviewer has noticed who used methods resting on a characteristic function prior to 1900. Strange to say, neither Thiesen nor Bruns seems to be aware of Hamilton's work.

With the turn of the century a marked change set in. F. Klein already in 1891 at a meeting of the *Naturforscher-Versammlung* had called attention to Hamilton's papers, and in his lectures in Göttingen, in the Summer Semester of that year he developed Mechanics as a kind of optics in an n-way space. In 1901 he called attention to the relation of Bruns's work to that of Hamilton. Klein says "I cannot close this short note without emphasizing the quite special importance of Hamilton's investigations in geometrical optics." The real source of Hamilton's profound dynamical theory lies in his optical views. Klein expresses the wish that Hamilton's optical papers may be made more easily accessible. Such a publication he declares would have not only historical value but would clarify and advance our ideas in many directions.

The first great advance, using Hamilton's optical methods, since Hamilton's last major paper *The Third Supplement* published in 1837 was made by K. Schwarzschild in his Untersuchungen zur geometrischen Optik in 1905, that is, after an interval of sixty-eight years. Contrary to the usual belief at that time, he showed how adaptable Hamilton's methods are to deduce practical formulae

in the design of optical instruments. As far as the reviewer is aware, Schwarzschild's paper has not been followed by others on the continent.

In England the situation is different. The last war made it clear to all that a great country could not allow itself to depend upon a foreign country for optical supplies. Immense efforts have been made to bring England back to the same relative level that it enjoyed in the days of Airy, Sir John Herschel, and Coddington. From all the reviewer can learn these efforts have been crowned with success. The feature most gratifying to him is their return to the method of Hamilton. We notice the splendid work of T. Smith[1] and G. C. Steward. It has commonly been thought that the methods of geometrical optics are useless in diffraction problems. Steward has shown[2] that the integral

$$\int e^{2\pi i(V+r)/\lambda} dS,$$

where V is a characteristic function and dS an element of surface may be used with marked success in treating aberration diffraction effects.

Not only have Hamilton's optical ideas been stimulating and fruitful today in the domain of pure optics but also in the new quantum mechanics. Schrödinger[3] who has been so successful in this field writes:—"Unfortunately this powerful and weighty circle of ideas of Hamilton has been robbed of its beautiful intuitive clothing in most of its modern presentations, as a superfluous scaffolding, in favor of a more colorless analytical treatment."

It is probably due to the new English school of theoretic and practical optics and the widespread interest in quantum mechanics that the Royal Irish Academy has incurred the heavy expense of publishing the Mathematical Papers of Sir William Rowan Hamilton. The moment is doubly opportune.

JAMES PIERPONT

Introduction to Higher Geometry. By W. C. Graustein. New York, The Macmillan Company, 1930. xv+486 pages.

The publication of this book would seem to be an event of major importance for the development of geometry in the United States. The qualifications of the author for writing such a book are eminent and well known. When a mathematician whose original researches of marked distinction in geometry have measurably advanced his science, and whose lectures in the classroom have become widely known for their clarity and thoroughness, announces a forthcoming volume from his pen, the result of his undertaking is to be awaited with impatience and received with enthusiasm. In this instance our expectations have been amply realized.

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[1] Cf. numerous papers "Transactions of the Optical Society" from volume 24 on.

[2] Trans. Roy. Soc. London, vol. 225, p. 131. Trans. Cambridge Phil. Soc., vol. 23, p. 236.

[3] Ann. d. Physik, vol. 79 (1926), p. 490.

This work of Graustein's seems to meet a definite need. Some of us who regularly teach *analytic projective geometry*, or *modern analytic geometry*, or *higher geometry*, or whatever else it may be called have been wishing for just such a book in English. Our students come to us with a knowledge of elementary metric analytic geometry in the plane and perhaps also in three-dimensional space. Some, but not all, of them have studied *synthetic projective geometry*. It is our task to prepare them for graduate work in *algebraic geometry*, *projective differential geometry*, and the like. In short, we are expected to lead them to some appreciation of developments in geometry as outlined by Klein in the Erlanger Programm. The book before us will be a very teachable text in these courses, and, moreover, will be just the thing to place in the hands of students who have not had the benefit of formal instruction in the subjects covered but who wish to read along these lines.

Examining the structure and contents of Graustein's book, we observe at once that it is built around the Erlanger Programm and contains nineteen chapters, of which only the last advances from geometry in the plane to geometry in space of three dimensions. It will not be necessary to attempt to outline the entire subject matter, but it does seem worth while to note the organization of the opening chapters. The first is entitled *Linear equations and linear dependence*, but it might properly have been called *Algebraic introduction*, since its purpose is to give the reader the essentials of higher algebra that he is going to need immediately. The second chapter is entitled *Geometrical introduction*, and may be passed over rapidly by readers who have studied synthetic projective geometry. In Chapter III homogeneous cartesian coordinates are introduced. In the next five chapters appear harmonic division, line coordinates, cross ratio, transformations, and metric geometry of the complex plane in natural and historical order. One-dimensional projective geometry is introduced in Chapter IX, and projective homogeneous coordinates in the plane are postponed to Chapter X.

The experience of the reviewer agrees with that of the author in dictating that it is most feasible pedagogically to base the treatment of our subject on the student's previous training in metric analytic geometry. The affine, projective, and other geometries should at first be presented to him couched in terms of his previous knowledge. Later on he can do what we might like to do in our first presentation, namely, organize each geometry on a postulational basis of its own, or at least he will then be in a position to appreciate a postulational presentation.

Experience differs as to details. For instance, the reviewer, after many trials and errors, has come to the conclusion that he can economically and effectively introduce general projective homogeneous point coordinates before attempting to consider homogeneous cartesian coordinates. The more general coordinate system seems to be more easily comprehended by the student than the more special cartesian system. The author, however, reverses this order, as has al-

ready been noted. Another instance of divergence of taste occurs in Chapter XVI. The reviewer is in the habit of calling the system of conics tangent to four given lines a *range* of conics, just as he calls the system of conics through four given points a *pencil* of conics, thus preserving the analogy between a *range* of points on a straight line and the dual *pencil* of lines through a point. The author does not seem to use the convenient terminology *range of conics*, but says *pencil of line conics*.

The author's treatment is thorough and painstaking with emphasis on fundamentals. The reasoning is as sound as it can be in a non-postulational exposition. Particularly to be commended is the sparing use of the idea usually expressed by the phrase "in general." And as a matter of language, when one says *in general* one ought to mean *in all special cases*, but what is usually meant is *except in one or more special cases*. When we wish to convey the latter meaning why can we not say "ordinarily" instead of "in general" and thus preserve for the phrase its true meaning?

There is an opulent wealth of material in the book. The number of available exercises at the ends of sections and chapters is amazing—more than 1000 by actual count!

The print is clear; the typography excellent. The attractiveness and beauty of the cover characterize a very appropriately chosen dress for a dignified volume.

While not daring to assume too ostensibly the role of a prophet, the reviewer does venture the prognostication that, in its influence on the mathematical teaching of first-year graduates in the colleges and universities of the United States, and as a book of reference, Graustein's *Introduction to Higher Geometry* will eventually take rank along with Bôcher's *Introduction to Higher Algebra* and Eisenhart's *Differential Geometry*.

ERNEST P. LANE

Die Relativitätstheorie. By Professor Dr. Ludwig Hopf. Berlin, Julius Springer, 1931. 147 pages.

This German book on the relativity theory, intended to be a popular treatment, turns out to be very scholarly. This is not to be taken as a disparagement of the work. On the contrary, the reviewer considers it as one of the best books on the subject offered for educated laymen. To gain from it a full understanding of the theory, however, one must possess a fair previous acquaintance with the theory's basic ideas. The reader already informed in this way about the essentials of the theory will find in Professor Hopf's book an excellent means of broadening his knowledge.

The work consists of three parts; preliminaries, special relativity theory, and general relativity theory. The first part, which occupies about one fifth of the whole book, is the best of the three. It is quite intelligible even to the beginner in the study of the relativity theory. At full length and with great

lucidity it discusses the transformation of the mechanistic "world picture" into the electromagnetic one in the decades preceding the advent of the theory. This account of preliminary developments, not covered satisfactorily in other popular books on the subject, is a valuable distinguishing feature of the work; and prepares the reader admirably for the study of the main theme of the book, the relativity theory, presented in the other two parts.

For the novice these will prove rather disappointing. The fundamentals of the theory, which interest him first of all, are not given in a manner intelligible for him. He will also be confronted with other puzzles. The book, for instance, states (p. 62) that "*a body moving with light velocity cannot be accelerated any more; no force of the world would suffice for this. It follows that the mass of a body must grow with the velocity.*" This conclusion does not at all follow from the premises. The beginner will certainly not see why the mass must grow with the velocity for the reason that a velocity greater than that of light is impossible. He will not be helped over the difficulty by the whole discussion of the connection between the mass and velocity of a body: for this discussion is not clear. Some important paragraphs are worded so obscurely for the tyro that he will hardly be able to understand them. Such paragraphs are those treating the "dependence of mass upon velocity" (pp. 61–62), "light propagation in the accelerated laboratory" (99–101), "rotating bodies" (107–109), "clocks in the gravitational field" (111–113), "two methods of time measurement" (117–119), etc.

Professor Hopf's book has therefore not invalidated the reviewer's contention (this MONTHLY, January, 1932) that the relativity theory has not yet found the ideal expositor. Nevertheless, considering the needs of the advanced student as well as those of the beginner, Professor Hopf's book represents a most valuable contribution to the literature of the relativity theory.

MAX TALMEY

PROBLEMS AND SOLUTIONS

EDITED *by* B. F. FINKEL, OTTO DUNKEL, *and* H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3543. *Proposed by L. M. Berkeley, New York City.*

A, B, C, and D are points on a sphere. Given all the parts (sides and angles)

of the spherical triangle ACD, and given also the angles ABC and ABD, find AB.

3544. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

The orthogonal sphere of four variable spheres, with fixed centers, whose radii remain proportional, describes a coaxial pencil.

3545. *Proposed by George Y. Sosnow, Newark, N. J.*

The equation

$$\frac{a_1}{x+b_1}+\frac{a_2}{x+b_2}+\frac{a_3}{x+b_3}+\cdots+\frac{a_n}{x+b_n}=0,$$

will be an identical equation if

$$\sum_{i=1}^{n}(a_i)=0,\quad \sum_{i=1}^{n}(a_ib_i)=0,\quad \sum_{i=1}^{n}(a_ib_i^2)=0,\cdots,\quad \sum_{i=1}^{n}(a_ib_i^{n-1})=0.$$

3546. *Proposed by W. H. Rasche, Virginia Polytechnic Inst.*

Given that r is the radius of a circle, S its plane, O its center, P any point in the plane of the circle, distant $d(<r)$ from the center; l is the line through P perpendicular to line OP which makes with plane S the angle $\alpha=\cos^{-1}(d/r)$; show that if the circle be rotated about l as an axis it will generate a torus whose meridian section is two equal non-intersecting circles of radius d, whose centers are equi-distant from axis l.

Note: A solution suitable for college freshmen not familiar with solid analytical geometry is desired.

3547. *Proposed by Martin Rosenman, Brooklyn, New York.*

Consider n points in a plane. Join these in any order to form a closed polygon. Repeat the operation on the n midpoints of the sides of the polygon thus formed, etc. Prove that the successive polygons converge to a point.

3548. *Proposed by J. B. Reynolds, Lehigh University.*

A coal chute delivers coal at a point at distances a and b from two vertical walls at right angles to each other. Find the volume of a conical pile of coal under the chute if its radius is r, $r>(a^2+b^2)^{1/2}$, and the angle of repose of coal is A.

3549. *Proposed by Dewey C. Duncan, University of California.*

Show that for just one set of positive constants $a_1, a_2, \cdots, a_{k-1}$, the equation

$$x^{2k+1}-(x-1)^3\prod_{i=1}^{k-1}(x-a_i)^2=0$$

has $k-1$ double roots. These double roots are necessarily positive and the remaining pair are complex.

3550. *Proposed by H. L. Fisher, Annandale, New Jersey.*

How much will be bored from a cubical block, edge, e, by boring seven holes through the block; one each through the centers of three pairs of opposite faces and one each along the four diagonals as axes, the radius of the auger being r?

3551. *Proposed by Eugene M. Berry, Lynchburg College.*

Find the equation of a curve whose evolute is the same as one of its involutes but not the same as the original curve.

SOLUTIONS

3486[1931, 171]. *Proposed by E. C. Kennedy, College of Mines, El Paso, Texas.*

Find a function of x such that the integral of that function between the limits of 0 and x is equal to the reciprocal of the original function.

Solution by Byron D. Roberts, Parsons College.

Let y be the required function of x, such that $\int_0^x y\,dx = y^{-1}$. Differentiating this equation we get a simple differential equation of the first order in which the variables separate in the form $y^{-3}dy = -dx$. The general solution of this equation is $y = (2x+c)^{-1/2}$; and for the required function we must set $c = 0$. The function is therefore $(2x)^{-1/2}$.

Also solved by H. T. R. Aude, A. G. Clark, Wei Liang Chow, J. D. Hill, V. F. Ivanoff, L. S. Johnston, J. H. Neelley, W. V. Parker, O. J. Ramler, H. D. Ruderman, F. Underwood, Kamcheung Woo, and the Proposer.

3487[1931, 227]. *Proposed by Vladimir F. Ivanoff, San Francisco, California.*

Prove the identities:

$$\frac{d^2y}{dx^2} + \frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3 \equiv 0, \tag{1}$$

$$\frac{d^3y}{dx^3}\left(\frac{dx}{dy}\right)^2 + \frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^2 + 3\frac{d^2y}{dx^2}\,\frac{d^2x}{dy^2} \equiv 0, \tag{2}$$

the derivatives d^ny/dx^n and d^nx/dy^n being based on the same functional relation $f(x, y) = 0$.

Solution by D. H. Richert, Bethel College

Since

$$\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy},$$

we have

$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(1 \Big/ \frac{dx}{dy}\right)\frac{dy}{dx} = -\frac{d^2x}{dy^2} \Big/ \left(\frac{dx}{dy}\right)^3$$

whence

$$\frac{d^2y}{dx^2} + \frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3 = 0,$$

which is (1).

Again, we have

$$\begin{aligned}
\frac{d^3y}{dx^3} &= -\frac{d}{dy}\left[\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3\right]\frac{dy}{dx} \\
&= -\left[\frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^3 + \frac{d^2x}{dy^2}\frac{d}{dy}\left(\frac{dy}{dx}\right)^3\right]\frac{dy}{dx} \\
&= -\left[\frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^3 + \frac{d^2x}{dy^2}\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)^3\frac{dx}{dy}\right]\frac{dy}{dx} \\
&= -\left[\frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^3 + \frac{d^2x}{dy^2}\cdot 3\left(\frac{dy}{dx}\right)^2\cdot\frac{d^2y}{dx^2}\frac{dx}{dy}\right]\frac{dy}{dx} \\
&= -\left[\frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^4 + 3\frac{d^2y}{dx^2}\cdot\frac{d^2x}{dy^2}\cdot\left(\frac{dy}{dx}\right)^2\right].
\end{aligned}$$

Whence, multiplying both sides of the equation by $(dx/dy)^2$ and transposing, we have

$$\frac{d^3y}{dx^3}\left(\frac{dx}{dy}\right)^2 + \frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^2 + 3\frac{d^2y}{dx^2}\frac{d^2x}{dy^2} = 0,$$

which is (2).

Also solved by A. G. Clark, W. R. Church, Louis Deutsch, J. D. Hill, J. F. G. Hicks, Theodore Lindquist, J. D. Leith, J. A. McLaughlin, Gertrude McCain, A. Pelletier, and E. P. Stark.

A Note by the Editors: These relations are derived on page 41 of vol. 1, Goursat-Hedrick, as we learn from a note by Norman Anning.

3488[1931, 228]. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, New York.*

Find the numbers m such that the square of each of the $\phi(m)$ numbers less than m and prime to it will be congruent to 1, modulo m.

Solution by J. D. Hill, University of California at Los Angeles

The numbers 1, 2, 3, and 4 are obvious solutions. In order to determine whether or not further solutions exist, we first note the following properties which such numbers necessarily possess:

(1) No odd number greater than 3 is a solution since 2^2 is not congruent to 1 for such a modulus.

(2) No square number greater than 4 is a solution. In view of the first property, assume m to be of the form $4x^2$ where x is greater than 1. But $4x^2$ cannot be a solution since $(2x-1)^2$ is not congruent to 1 for this modulus, although $2x-1$ and $4x^2$ are relatively prime.

(3) The number m is not a solution unless $m-1$ is a prime. For if $m-1$ is not a prime it has a prime factor p less than the square root of m, whereas p^2 is obviously not congruent to 1, modulo m.

(4) The number m to be a solution must be divisible by the product of all the primes less than its own square root. For if q is any prime less than the square root of m, which does not divide m, then it is evident that q^2 is not congruent to 1, modulo m.

For simplicity we first seek solutions among the integers 5 to 49 inclusive. To begin with, the odd numbers in the interval are discarded and after these the two squares, 16 and 36. Of the remaining even integers 10, 22, 26, 28, 34, 40, and 46 fail to satisfy (3), while 14, 20, 32, 38, 42, 44 and 48 do not meet condition (4). This leaves 6, 8, 12, 18, 24, and 30 as possible solutions, and of these numbers only 18 and 30 fail to fulfill the conditions of the problem.

That 1, 2, 3, 4, 6, 8, 12, and 24 are the only solutions of this problem may be proved as follows: Denote by P_n the product of all the primes equal to or less than n, and consider the numbers m defined by the inequality $n^2<m<(n+1)^2$, where $n\geqq 7$. By the property (4), m must be divisible by P_n in order to be a solution. But by the well known theorem that the sum of the natural logarithms of all primes $\leqq n$ equals n asymptotically,[1] it is easily shown that $P_n>(n+1)^2$, for $n\geqq 7$. Therefore no number greater than 49 could possibly be a solution, and we have the compact result that the conditions of the problem are satisfied by and only by the positive integral factors of 24.

Also solved by A. G. Clark, W. R. Church, Emma Lehmer, D. E. Rutherford, E. P. Stark, and F. L. Wilmer.

3489[1931, 288]. *Proposed by J. G. Deutsch, Columbia University.*

Given four points in a plane, no three of which are collinear, what is the necessary and sufficient condition that there exist a square containing the given points on its perimeter such that no two points are on the same side?

Solution by Roger A. Johnson, Brooklyn College

The problem is not well stated, for two reasons: first, there is no specification as to the order of the given points, and second, because the restriction that the points be not collinear is unnecessary. We may propose the problem in the following alternative form:

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[1] See Dickson's History, Vol. 1, page 439.

Given four points in a plane in definite order, to construct a square whose sides shall pass successively through these four points.

The result then is: if A, B, C, D are the points, so that A and C are to lie on opposite sides of the required square, B and D on the other pair of opposite sides, there always are two solutions except when AC and BD are perpendicular. In this exceptional case, if AC and BD are unequal there is no solution; if they are equal, there are infinitely many solutions.

Analysis. Let A, B, C, D be the given points; draw a rectangle $PQRS$ so that PQ, QR, RS, SP pass respectively through A, B, C, D. (It is sufficient to draw any parallels through A and C, and perpendiculars to them through B and D.) Let AH be perpendicular to RS, and DK to QR. In order that the rectangle may be a square, it is necessary and sufficient that $AH=DK$. Therefore, if the right triangle DBK were turned through 90° and D superposed on A, K would fall on H, and B, H, C would be collinear. Hence the construction.

Construction. Through A draw AE equal and perpendicular to DB. Then CE is the direction of the side of the required square which passes through C; and the square is at once constructed by drawing a line through A parallel to CE, and lines through B and D perpendicular to it.

Since AE can be drawn in either of two opposite directions, there are necessarily two solutions. The analysis and the construction work equally well when three or even four of the points are collinear. A failing case appears when AC and BD are perpendicular and unequal in length, for then AE falls on AC. If $AC=AE=BD$, any direction may be taken; i.e., any rectangle circumscribed about $ABCD$ is a square if AC and BD are perpendicular and equal. It can be proved at once that if A, B, C, D lie on the sides of a square and AC is perpendicular to BD, then $AC=BD$. Hence, *a priori* the case that AC and BD are perpendicular and unequal is impossible.

Also solved by E. B. Escott, H. E. Stelson, and F. Underwood.

3490[1931, 288]. *Proposed by H. E. Stelson, Kent State College, Kent, Ohio.*

Sanford's *History of Mathematics*, page 264, tells of Pappus's (c. 300) method of trisecting an angle by the use of conics. He placed the angle in question with its vertex at the center of a circle. He then trisected the subtended chord AB and constructed a hyperbola having AA' two thirds of the chord as its transverse axis and $\sqrt{3}$ times the transverse axis for the conjugate one.

Prove analytically that points joined to the ends of the chord such that one of the angles thus formed is double the other lie on the above hyperbola.

Solution by Margaret M. Young, Brooklyn College of the City of New York

Let the axes be taken so that the coordinates of A, A', B are, respectively, $(a, 0)$, $(-a, 0)$, $(-2a, 0)$; and let $P(x, y)$ be a point chosen so that $\angle ABP = 2\angle BAP$. By the law of sines

$$\frac{BP}{AP} = \frac{\sin A}{\sin 2A} = \frac{1}{2\cos A}, \tag{1}$$

where

$$BP = [(x+2a)^2 + y^2]^{1/2}, \quad AP = [(x-a)^2 + y^2]^{1/2}, \tag{2}$$

$$\cos A = (a-x)/AP.$$

From (1) and (2) we obtain after certain reductions the equation of the locus of P

$$4(x-a)^2(x+2a)^2 = [(x-a)^2 - y^2]^2. \tag{3}$$

This equation is equivalent to two equations, one of which is

$$x^2 + 4ax + y^2 - 5a^2 = 0. \tag{4}$$

This equation (4) results from a triangle ABP where $180° - \angle ABP = 2\angle BAP$, and therefore $BA = BP$. This case is included in the analytical work, but is to be rejected. The desired equation is then the other part of (3)

$$3x^2 - y^2 - 3a^2 = 0. \tag{5}$$

It is an hyperbola of the kind described in the problem, and the branch through A' is the desired locus.

A Note by Otto Dunkel: Since it may be foreseen that the law of sines introduces an extraneous equation of a circle, it would be simpler to replace this law by the relation $\tan B = 2\tan A(1-\tan^2 A)^{-1}$, where $\tan B = y/(x+2a)$, $\tan A = y/(a-x)$. After rejecting a factor y, we have the desired equation.

It should be observed that the construction given by Pappus requires the proof of the converse of the theorem at the end of the problem. This proof may be obtained by altering the form of equation (5) so as to introduce $\tan B$ and $\tan A$; in other words we simply reverse the order of the above proof.

If BA is prolonged to D so that $BD = 4a$, the branch through A' of the hyperbola of Pappus may be defined as the locus of P such that $DP = BP + 2a$. The proof may be now given by methods in use in the days of Pappus. Let M be the projection of P on BA, then $\overline{DM}^2 - \overline{MB}^2 = \overline{DP}^2 - \overline{BP}^2$. After factoring and using the above definition of P, this reduces to $2(DM - MB) = DP + BP = 2(BP + a)$. Hence $AM = DM - a = MB + BP$. Produce AB to Q so that $BQ = BP$, then $AM = MQ$, and the triangle AQP is isosceles. Hence $\angle ABP = 2\angle BQP = 2\angle BAP$.

Also solved by J. E. Burnam, A. Pelletier, O. J. Ramler, Wallace Smith, and F. Underwood.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

A professorship of mathematics has been established at Yale University in honor of Josiah Willard Gibbs. Professor E. W. Brown, previously Stirling professor of mathematics at the University, has been made the first incumbent of this new professorship.

Professor Oswald Veblen, of Princeton University, is on leave of absence during the second semester of the current academic year. On December 18, he lectured at Duke University on *Spinor analysis;* on January 8, 11, and 12, he lectured at the Rice Institute on the topics *The modern approach to elementary geometry*, *Analysis situs*, and *The new differential geometry;* and on January 25, he lectured at The University of California at Los Angeles on *Projective relativity*. In February he will go to Europe, and will lecture at the universities of Göttingen, Hamburg, and Berlin.

Professor H. N. Russell, of the department of astronomy of Princeton University, has been elected president of the American Philosophical Society.

The following have been appointed associate editors of the Transactions of the American Mathematical Society: E. P. Lane, Marston Morse, M. H. Stone.

The Scientific and Research Institute of Mathematics and Mechanics of the University of Moscow, has awarded to L. S. Pontrjagin, member of the Institute, a prize of 750 roubles for his paper "Über stätige algebraische Körper" which has just appeared in the Annals of Mathematics.

Assistant Professor J. A. Nyswander, of the University of Michigan, has been promoted to an associate professorship.

Mr. D. E. Whitford has been promoted to an assistant professorship at the Brooklyn Polytechnic Institute.

The following courses in mathematics are announced for the summer 1932:

University of California at Los Angeles, July 1 to August 10. By Professor Paul H. Daus: Foundations of arithmetic.

University of Maine, July 5 to August 12. In addition to the usual elementary work, the following advanced courses are offered. By Associate Professor Bryan: Teacher's course. By Associate Professor Jordan: Practical astronomy. By Professor Willard: Theory of functions of a complex variable.

Ohio State University, June 21 to September 2. In addition to the usual courses in trigonometry, analytic geometry, and the calculus, the following advanced courses will be offered. By Professor C. C. Morris: Theory of probability; Differential equations. By Professor J. H. Weaver: Projective geometry; Differential geometry. By Professor Lincoln LaPaz: Calculus of variations; Tensor analysis. By Dr. H. P. Theilman: Integral equations.

Stanford University, June 23 to September 3. In addition to the usual courses in calculus and differential equations, the following courses are offered. By Professor G. T. Whyburn of Johns Hopkins University: Theory of functions of a real variable; Foundations of geometry; Point-Set theory.

On February 28, there died at Providence, R. I., Dr. Arnold Buffum Chace, Chancellor (President of the Corporation) of Brown University. His edition of the Rhind Mathematical Papyrus is probably the most elaborate publication ever made of any ancient mathematical manuscript, and is certainly one of the most scholarly. For more than a half century he was a member of the Brown Corporation, and during most of this time he held either the office of treasurer or that of chancellor. He was educated at Brown and did advanced work in chemistry in Europe.

Miss Rose B. Wood, of Greenville Woman's College, died July 28, 1931. She was a charter member of the Association.

The announcement has been received of the death of Professor Doctor Heinrich Wieleitner, Oberstudiendirektor am neuen Realgymnasium und Honorarprofessor an der Universität, München, on December 27, 1931. Professor Wieleitner was well-known to all students in the history of mathematics. He was a serious scholar and a clear and accurate writer. His loss will be felt in this country as well as in Munich, where he worked for so many years.

It is known that Edouard Combescure (1824–1889), a docteur es-sciences of Paris (1858) and professor of Astronomy at Montpellier (1868) spent some time in the United States. It is desired to know if he was connected with any university here, and if he published any scientific papers while in this country. Any particulars relating to these matters will be welcome. Please send any information to Professor David Eugene Smith, 501 West 120th Street, New York.

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teachers' agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

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(Signed) M. C. Erwin, Head Math Dept., Reynolds High School, Reynolds, Ind.

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The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The prize will be awarded hereafter every three years. The last award was in December, 1929, to Professor T. H. Hildebrandt. The next award will be in December, 1932, for the period 1929-1931.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Sixteenth Summer Meeting of the Association, Los Angeles, Calif., Aug. 29-30, 1932.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1932 and reported to the Secretary.

ILLINOIS, Urbana, May 6-7.
INDIANA, Indianapolis, May 6-7.
IOWA, Cedar Falls, April 29-30.
KANSAS, Topeka, Feb. 13.
KENTUCKY.
LOUISIANA-MISSISSIPPI, Oxford, Miss., March 11-12.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7.
MICHIGAN, Ann Arbor, March 19.
MINNESOTA.
MISSOURI.
NEBRASKA.
OHIO, Columbus, Ohio, April 7.
PHILADELPHIA, Philadelphia, Pa., Nov. 26.
ROCKY MOUNTAIN, Laramie, Wyo., April 15-16.
SOUTHEASTERN, Gainesville, Fla., Mar. 18-19.
SOUTHERN CALIFORNIA, San Diego, March 26.
TEXAS, Austin, Jan. 30.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS.
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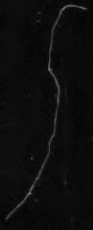

THE SEVENTH ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The seventh annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Natchitoches, La., on March 13 and 14, 1931, in connection with the annual meetings of the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics and the Louisiana Academy of Sciences. The session Friday afternoon was devoted largely to Section matters, Professor I. C. Nichols, chairman of the Section, presiding.

The attendance was well over one hundred, including the following fifteen members of the Association: Nola L. Anderson, T. A. Bickerstaff, Leora Blair, H. E. Buchanan, C. G. Jaeger, C. G. Killen, A. C. Maddox, B. E. Mitchell, I. C. Nichols, R. L. O'Quinn, M. C. Rhodes, S. T. Sanders, C. D. Smith, H. L. Smith, W. P. Webber; and two official representatives of institutional members: J. P. Cole and J. A. Hardin.

The following officers were chosen for the coming year: Chairman, A. C. Maddox, Louisiana State Normal College; Vice-chairmen, C. D. Smith, Mississippi A. and M. College and J. P. Cole, Louisiana Polytechnic Institute; Secretary-Treasurer, Nola L. Anderson, Sophie Newcomb College. The University of Mississippi was selected as the place for the 1932 annual meeting.

Three attractive features of the convention were the joint tea in the social room of the Normal College, served by the local branch of the American Association of University Women; the joint illustrated lecture by Doctor Frans Blom of Tulane University on "Middle American research," Professor George Williamson presiding; and the joint banquet in the College dining hall, Doctor Nichols presiding as toastmaster. About two hundred persons attended the banquet. Excellent music was furnished by the college band and glee clubs.

At the Friday session of the Section the following papers were presented:

1. "Ancient Egyptian problems" by Professor E. M. Shirley, Louisiana Polytechnic Institute, by invitation.
2. "The construction of the helium atom" by Professor H. E. Buchanan, Tulane University.
3. "The rise of a new geometry" by Professor C. D. Smith, Mississippi A. and M. College.
4. "Trigonometry in a space of n dimensions" by Professor Nola L. Anderson, Sophie Newcomb College.
5. "Certain partial derived functions" by F. A. Richey, Mandeville, La., by invitation.
6. "A certain theorem concerning two triangles" by Professor C. G. Jaeger, Tulane University.
7. "Sextantal analysis" by Professor B. E. Mitchell, Millsaps College.
8. "Maya numbers" by Professor Frans Blom, Tulane University, by invitation.